SOLVED PAPER

M.Sc. MATHEMATICS (Part II)

ADVANCE ANALYSIS

SARFRAZ PHOTOCOPY

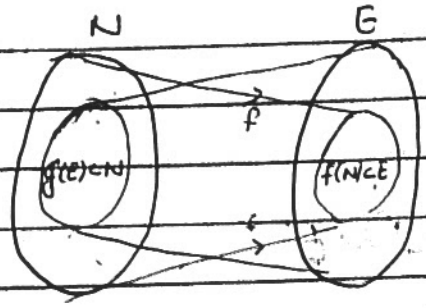
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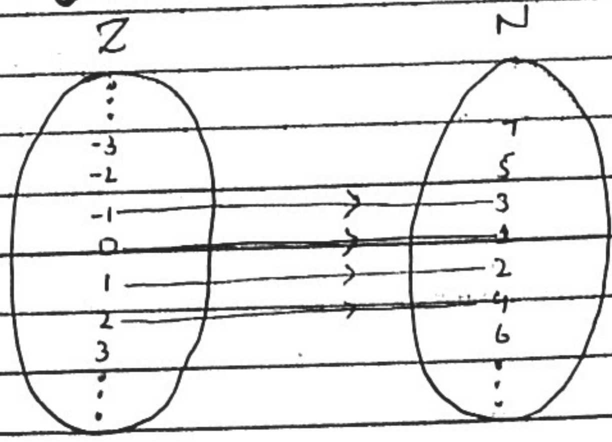
ALI KHAN: 0324-4863102

Sec I



Example A|2013
Show that $N \sim Z$

Proof.



Define
$f: Z \to N$

Inverse $x = \frac{y+1}{2}$

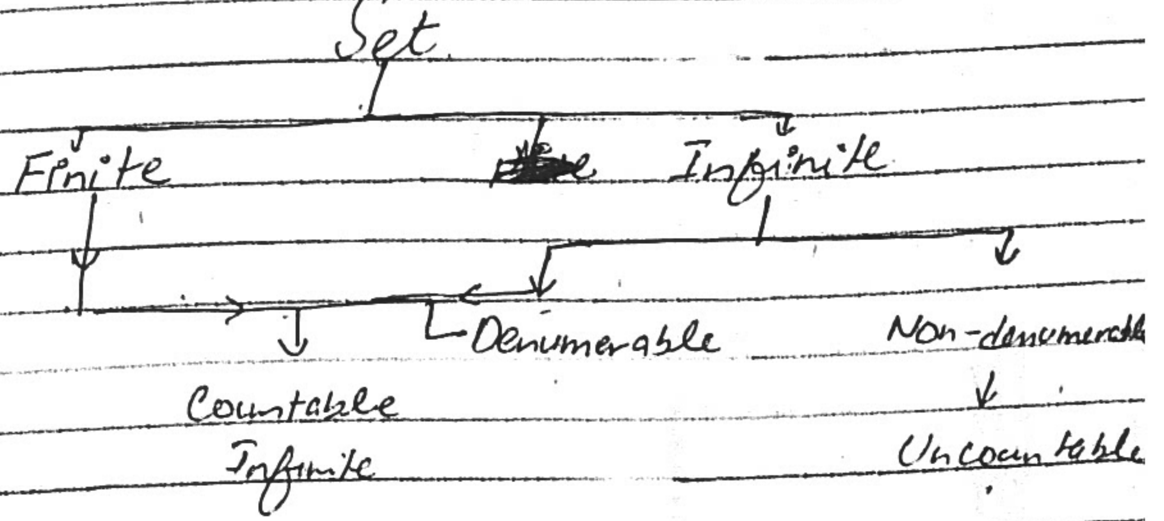
$$f(x) = \begin{cases} 2x & x \in Z^+ \\ 1-2x & x \in \{0\} \cup Z^- \end{cases}$$

f is bijection from $Z \to N$.

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Uncountable Set.

A set which is neither finite nor denumerable is called uncountable set.



$f(n) = 1/n$
$A = \{1, \frac{1}{2}, \dots \frac{1}{n}, \dots\}$
$A \sim N$.

Theorem: A/2011 Q1(ii)

A subset of a denumerable set is either finite or denumerable.

Proof.

we arrange the numbers in terms seq.

Consider any denumerable set A. Then elements of A can be arranged as

$$A = \{a_1, a_2, \dots, a_n, \dots\} \quad \text{--- (1)}$$

ed
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nerable.

Let B be a subset of A.

Case I: If $B = \phi$, then B is finite.

Case II If $B \neq \phi$

Let 'b_1' be the first element in sequence (1) such that $b_1 \in B$.

Next, let 'b_2' be the first element which follows 'b_1' and so on.

Then

$$B = \{ b_1, b_2, \ldots b_{n_k}, \ldots \} \quad \text{--- (2)}$$

If the sequence in (2) ends then B is finite otherwise B is infinitely countable (denomerable) set.

Theorem. Every infinite set A contains a subset which is denumerable.

Proof. Let A be an infinite set and $P(A)$ be the power set of A. Let

$f: P(A) \to A$ be a choice function. ~~(defined by)~~

Consider the following sequence

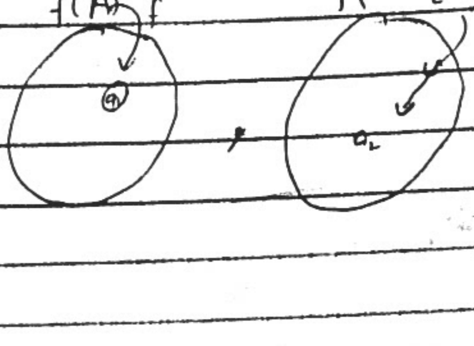
$$a_1 = f(A)$$

$$a_2 = f(A \setminus \{a_1\})$$

$$a_3 = f(A \setminus \{a_1, a_2\})$$

$$\vdots$$

$$a_n = f(A \setminus \{a_1, \ldots, a_{n-1}\}).$$



... so on.

Seq. of distinct elements is denumerable.

Since A is infinite, so

$$A \setminus \{a_1, a_2, \ldots a_{n-1}\} \neq \phi \quad , \forall n \in N$$

For if $A \setminus \{a_1 \ldots a_{n-1}\} = \phi$, A is finite.

Furthermore, f being choice function chooses exactly one element from each subset of A.

i.e, $a_i \neq a_j$ for $i < j$.

Therefore, $\{a_n\}$, being infinite sequence of distinct element, is denumerable.

اگر کسی set کو ہم Seq کی Arrangement میں لکھ سکیں تو وہ سیٹ denumerable ہوتا ہے۔

Theorem. A/2011 Show that every infinite subset of a denumerable set is denumerable.

Any subset of a countable set is countable.

Proof. Let A be a subset of countable set B. Then B is either finite or denumerable.

We show that A is countable.

Case I: B is finite.

When B is finite then A is also finite. Therefore A is countable.

Case II. B is denumerable.

When B is denumerable then A is also denumerable because a subset of a denumerable set is either finite or denumerable.

A/2013
A/2017, Let $P=\{1,2,3,\ldots\}$ be the set of +ve integers. Then show that $P\times P$ is denumerable.

imp

Example. Prove that $N\times N$ is denumerable.

or $N\times N\sim N$
or $N\times N$ is countable

A/2012. Prove that $N\times N$ is countable and deduce that ...

ALI HAIDER

Proof. We know that every natural number 'n' can be expressed as

$$n=2^{l-1}(2m-1),\quad l,m\in N. \quad (1)$$

It is natural to define a function

$$f: N\times N \to N$$

defined by,

$$f((l,m)) = 2^{l-1}(2m-1)=n,\quad l,m\in N.$$

Note that f is one-to-one

$$2^{l-1}(2m-1) = 2^{p-1}(2q-1)$$

$$2^{l-p} = \frac{2q-1}{2m-1}$$

The above equality holds only if each side eq to 1.

$2^{l-p} = 1$; $\frac{2q-1}{2m-1} = 1$

$\Rightarrow l-p = 0$

$\Rightarrow l = p$ $\Rightarrow m = q$

Now define

$g: N \to N \times N$. defined by

$g(n) = (n, 1)$

Then g is one-to-one

By Shrö:der - Berstein - Theorem
$N \times N \sim N$.

Prob. 6.10, pg 155.

A/2012

Theorem: A countable union union of finite sets is countable.

Deduce that $n_0 + n_0 + \cdots = a$, where n_0 is the cardinal number of a finite set, a is the cardinal number of a countable set.

$A_1 \cap A_3 = S_1 \cap (S_3 \setminus S_2)$ ✓ $A_1 \cap A_2 = S_1 \cap S_2 \setminus S_1$

$=$ 9 $= \varphi$

Let $C = \{S_i : i \in \mathbb{N}\}$ be a countable collection of finite sets, and let

$C = \cup_i S_i$

Case I

If $C = \varphi$, then C is countable.

Case II

Suppose $C \neq \varphi$.

Define

$A_1 = S_1$,

$A_2 = S_2 \setminus S_1$,

$A_3 = S_3 \setminus (S_1 \cup S_2)$ and so on.

Then the set A_i are finite and pair wise disjoint.

Let

$A_1 = \{a_{11}, a_{12}, \ldots, a_{1n_1}\}$,

$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots, a_{2n_2}\}, \ldots$

Then the union

$B = \cup_i A_i = \cup S_i$ can be written as a sequence as follows

$B = \{a_{11}, a_{12}, \ldots, a_{1n_1}, a_{21}, a_{22}, \ldots, a_{2n_2}, \ldots\}$

Define $f : B \to \mathbb{N}$ such that

$f(a_{ij}) = n_1 + n_2 + \cdots + n_{i-1} + j$

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$$\left(\begin{array}{l} f(a_{13}) = n_1 + n_2 + n_{3-1+1} = n_1+n_2+n_3 \\ f(a_{24}) = n_1 + n_2 + n_{4-1+2} = n_1+n_2+n_3 \end{array} \right)$$ Book wala wrong

10

$$f(a_{ij}) = \sum_{k=i}^{i+j-1} n_k = n_i + n_{i+1} + \cdots + n_{i+j-1}$$

$$f(a_{13}) = \sum_{k=1}^{3} n_k = n_1+n_2+n_3$$

$$f(a_{24}) = \sum_{k=2}^{3} n_k = n_2+n_3.$$

Hence f is a ~~bijection~~ one-to-one.

Next let $g: N \to B$.

$$B = \{a_{11}, a_{12}, \ldots a_{1n_1}, a_{21}, a_{22}, \ldots\}$$

$$N = \{1, 2, 3, \ldots\}$$ پہلا element لیں گے

$g(n) = a_{n2}$, $g(N) \subseteq B$.

g is also 1-to-1.

By Schröder Berstein Theorem.

$B \sim N$.

But $B = \bigcup_i A_i = \bigcup_i S_i$

$\Rightarrow C = \bigcup_i S_i \sim N$.

$\Rightarrow C \sim N$

25th Aug 2017

A/2017 [A/2010 · Q no 1(b)]

Theorem: Let $A_1, A_2, \ldots, A_n, \ldots$ be a sequence of pairwise disjoint denumerable sets. Then $D = \bigcup_i A_i$ is denumerable

OR

Union of countable family of countable sets is countable.

Proof.

Case I.

Let each A_i be finite set then we know that countable union of finite sets is countable/denumerable.

Case II.

Let each A_i be a denumerable set then each elements of A_i's can be arranged as

$A_1 = a_{11} \quad a_{12} \quad a_{13} \ldots \quad a_{1n} \ldots$

$A_2 = a_{21} \quad a_{22} \quad a_{23} \ldots \quad a_{24} \ldots$

$\vdots$

Let us define sets

$B_1 = \{a_{11}\}$

$B_2 = \{a_{12}, a_{21}\}$

$B_3 = \{a_{13}, a_{22}, a_{31}\}$

$\vdots$

$B_n = \{a_{ij} \mid i+j = n+1\}$ and so on.

Note that each B_i is finite. Thus countable union of B_i's is also countable.

i.e., $\bigcup_i B_i$ is countable

Also

$$\bigcup_i B_i = D = \bigcup_i A_i$$

Hence

$\bigcup_i A_i$ is countable/denumerable

Example

Show that $N \times N$ is countable/denumerable.

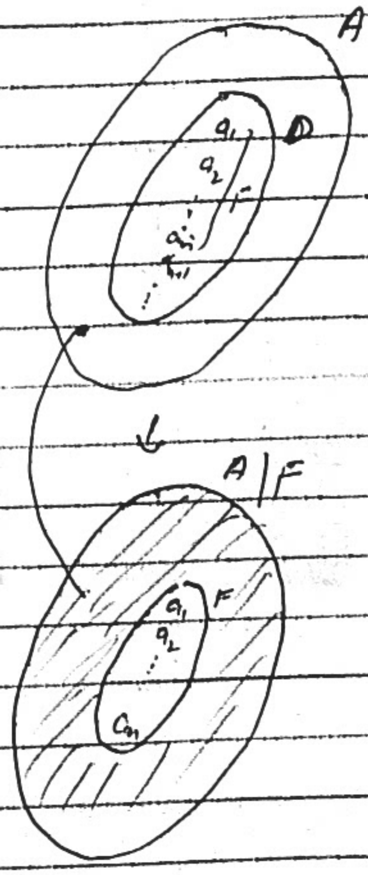
Proof Alt. Method.

We can write elements of $N \times N$ as

Define a function
$f: A \to A|F$ such that

$$f(a) = \begin{cases} a \ ; & a \neq a_k \ (a \notin D) \\ a_{k+n}, & a = a_k \ (a \in D) \end{cases}$$

Then f is a bijection.
Hence
$A \sim A|F$.



A/2015

Theorem: Suppose A is an uncountable set and B is a denumerable subset of A. Then

$$A|B \sim A$$

OR

Removing a denumerable set from an uncountable set does not change its size.

Proof.

Given that B is a denumerable set. So, we can write elements of B as

$$B = \{ b_1, b_2, \ldots, b_n, \ldots \}.$$

Also given that A is uncountable and hence infinite. Therefore we can find (extract) a denumerable subset D of A given by

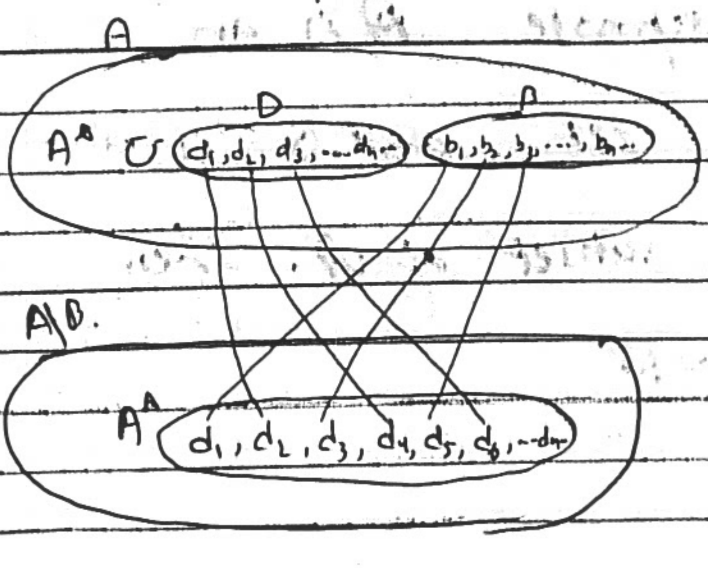
$$D = \{ d_1, d_2, \ldots d_n, \ldots \} \qquad D \in A \setminus B$$

Then the set A can be expressed as

$$A = A^a \cup B \cup D.$$

where

$$A^a = A \setminus (B \cup D).$$



Let us define a function f

$$f : A \longrightarrow A|B.$$

$$f(a) = \begin{cases} a & ; a \in A^{*} \\ d_{2k} & ; a = d_k \ (a \in D) \\ d_{2k-1} & ; a = b_k \ (a \in B). \end{cases}$$

Then f is a bijection from A to $A|B$.
Hence $A \sim A|B$.

Example.

A/2014, A/2013

The unit interval $[0,1]$ is uncountable (or non-denumerable).
and deduce that $[a,b]$ is non-denomerable.

Proof. Suppose $I = [0,1]$ be unit interval.
On contrary suppose that $I = [0,1]$ is denumerable. Then its elements can be arranged in an infinite sequence of distinct elements.

$$x_1 = 0.\,a_{11}\,a_{12}\,a_{13} \cdots a_{1n} \cdots$$

$$x_2 = 0.\,a_{21}\,a_{22}\,a_{23} \cdots a_{2n} \cdots$$

$$x_3 = 0.\,a_{31}\,a_{32}\,a_{33} \cdots a_{3n} \cdots$$

$$\vdots$$

$$x_n = 0.\,a_{n1}\,a_{n2}\,a_{n3} \cdots a_{nn} \cdots$$

$\frac{1}{2} = 0.5$
$= 0.499999\ldots$

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$\frac{22}{7} \in Q$
$\pi \in Q'$

$\pi \approx \frac{22}{7}$

where each element x_i can be written in the form of an infinite decimal and

$a_i \in \{0, 1, 2, \ldots, 9\}$

Let us consider (construct) a real number $y \in I = [0,1]$, y as

$y = 0.b_1 b_2 b_3 \ldots b_n \ldots \in I$, $b_i \in \{0, 1, \ldots 9\}$

Such that

$b_1 \neq a_{11}$, $b_2 \neq a_{22}$, ..., $b_n \neq a_{nn} \ldots$

Then by its contruction, $y \neq x_n$, $\forall n \in N$.
It implies that $y \notin I$, a contradiction to the fact that $y \in I$.
Hence our supposition that $I = [0,1]$ is denumerable is false.

Similarly if $y = a.b_1 b_2 \ldots b_n \in I = [a, b]$
$\therefore y \notin I = [a,b]$
Hence $I = [a,b]$ must be uncountable.

A/2016, A/2011

Example: Show that Q is denumerable.

Proof. $Q = Q^+ \cup \{0\} \cup Q^-$
First we show that Q^+ is denumerable.
Note that a function $f: Q^+ \to N \times N$ defined by

$$f\left(\frac{p}{q}\right) = (p, q), \quad (p, q) = 1$$

is one-to-one and onto, where

$\frac{22}{7} \in Q$ Denumerable

$\pi \in Q'$ if A an B [7] then $A \sim B$.

$\mathbb{R} \sim (-1, 1) \sim [0, 1]$.

$\pi \approx \frac{22}{7}$

then

[0, 1],

p and q are relatively prime positive integers.

$\Rightarrow Q^+ \sim N \times N \sim N$.

$\Rightarrow Q^+$ is countable

Also $g: Q^+ \rightarrow Q^-$ defined by $g(x) = -x$, is a bijection.

$\times N$

Next $\{0\}$, being a finite set, is countable set.

$\in I = [a, b]$

$Q = Q^+ \cup \{0\} \cup Q^-$, being a union of countable sets, is countable.

at the

$\mathbb{R} = Q \cup Q'$

A/2016

Show that Q' set of irrationals, is uncountable.

Proof.

Since $\mathbb{R}$ is uncountable and Q is a denumerable subset of $\mathbb{R}$, so

$Q' = \mathbb{R} | Q \approx \mathbb{R}$

$N \times N$

$Q' \sim \mathbb{R}$

Q', being equivalent to uncountable set $\mathbb{R}$ is also uncountable

$\mathbb{R} \supseteq (-1,1) \supseteq [0,1]$

6-28, 6-29. ✓ 18

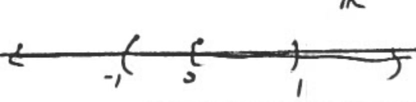
imp A/2015, A/2013.

Schröder Bernstein Theorem. (Alt. form)

Let X, Y, X_1 be the sets such that $X \supseteq Y \supseteq X_1$ with $X \sim X_1$

Then

$X \sim Y$

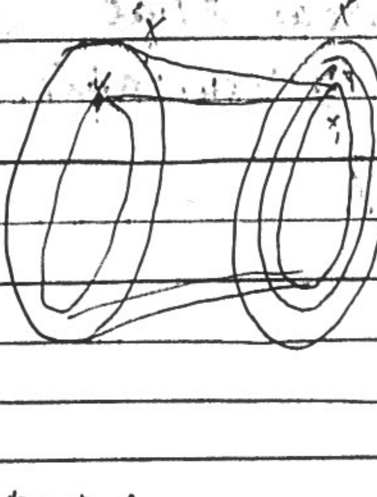


Proof.

Given that $X \sim X_1$. There is a bijection $f: X \to X_1$

Since Y is contained in X, therefore $f|_Y$ (restriction of Y) is one-to-one.

~~Let~~ From now on, we use f instead of $f|_Y$



Let $f(Y) = Y_1$.

$f: Y \to Y_1$ is a bijection.

$X \supseteq Y \supseteq X_1 \supseteq Y_1$

Using the similar reasoning

$f: X_1 \to X_2 = f(X_1)$ is a bijection

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optional step

X

$$X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2 \supseteq Y_2 \supseteq X_3 \supseteq Y_3 \cdots$$

i.e., $X_1 \sim X_2$

continuing in this way

$$X \sim X_1 \sim X_2 \sim X_3 \sim \cdots$$

$$Y \sim Y_1 \sim Y_2 \sim Y_3 \sim \cdots$$

Accordingly, $X_1, X_2, X_3, \ldots, X_n, \ldots$

and $Y_1, Y_2, \ldots Y_n, \ldots$ are (separately)

equivalent.

In general

$f: X_k \to X_{k+1}$ then $X_k \setminus Y_k \sim X_{k+1} \setminus Y_{k+1}$

$f: Y_k \to Y_{k+1}$ are bijection.

Note that we can write.

X and Y

$X_1 \setminus Y_1 \xrightarrow{f} X_2 \setminus Y_2$

$$X = (X \setminus Y) \cup (Y \setminus X_1) \cup (X_1 \setminus Y_1) \cup \ldots \cup B$$

$$Y = (Y \setminus X_1) \cup (X_1 \setminus Y_1) \cup (Y_1 \setminus X_2) \ldots \cup B$$

where

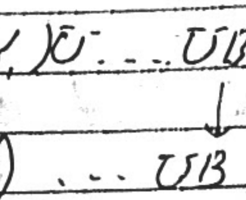
$$B = X \cap Y \cap X_1 \cap Y_1 \cap \ldots$$

optional step

$$X = (X \setminus Y) \cup (X \cap Y)$$

$$= (X \setminus Y) \cup \left[(X \cap Y) \setminus X_1 \cup (X \cap Y \cap X_1) \right]$$

$$= (X \setminus Y)$$



✓

28 Aug 2017

Let us define a mapping

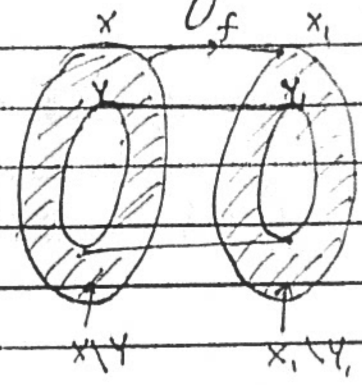
$g: X \to Y$. defined by.

$$g(x) = \begin{cases} x = I(x) & x \in Y_k \setminus X_{k+1} \text{ or } x \in B. \\ f(x) & x \in X_k \setminus Y_k \end{cases}$$

$\uparrow X_k \sim X_{k+1}$ via f.
$Y_k \sim Y_{k+1}$ via f.

This is also a bijection from X to Y.

$X \sim Y$.



Ex. Set of all polynomials P with integral coefficient, is denumerable.

Proof. The Polynomial of degree 'm' can be written as.

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

$a_i \in \mathbb{Z}, \forall i = 1, 2, \ldots m.$

Let $P(n, m)$ be a set all polynomials of degree m such that

✓

Cantor's Theorem.

A/2015 A/2012 A/2014

For any set A, we have

$|A| < |P(A)|$ or

For any cardinal number α, which have we $\alpha < 2^{\alpha}$.

$|N| < |P(N)|$

$\aleph_0 < 2^{\aleph_0}$

Proof. Let A be any set and P(A) be power set of A. Define a function $g : A \to P(A)$ defined by, good element.

$g(a) = \{a\}$ is one-to-one or injective.

let $a_1 \neq a_2$ $\Rightarrow g(a_1) \neq g(a_2)$ $\Rightarrow \{a_1\} \neq \{a_2\}$

$\Rightarrow |A| \leq |P(A)|$.

Next we need to prove that

~~also~~, $|A| \neq |P(A)|$.

On contrary, let us suppose

$|A| = |P(A)|$

$\Rightarrow$ there is a bijection

$f : A \to P(A)$

Let 'x' be a "bad" element if $x \notin f(x)$.

So, let B be the collection of all such bad elements in A set.

i.e $B = \{x : x \in A;\ x \notin f(x)\}$.

$\Rightarrow$ $B \subseteq A$ and $B \in P(A)$.

Since f is onto function, therefore there exit $b \in A$ such that

$$f(b) = B. \quad \text{—} (I)$$

Case I. If b is bad. $\Rightarrow b \in B$

$\Rightarrow$ $b \notin f(b) = B$.

(I) $\Rightarrow$ $b \notin B$. , contradiction

Case II if b is good element.

$\Rightarrow b \notin B$

$\Rightarrow b \in f(b) = B$.

$\Rightarrow$ $b \in B$. , a contradiction

Hence our supposition was wrong that $|A| = |P(A)|$

Thus $|A| \neq |P(A)|$.

Hence

$$|A| < |P(A)|$$

A/2014 A|2014

Theorem. Prove that

$$c = 2^{\aleph_0}$$

$|P(Q)| = c.$
$|P(N)| = c.$

Proof. Let $\mathbb{R}$ be a set of real numbers and $P(Q)$ be the power set of the set of rational numbers, Q.

Let f be the function defined by

$$f : \mathbb{R} \longrightarrow P(Q)$$

$$f(a) = \{x \mid x \in Q : x < a\} \subseteq Q \quad \in P(Q)$$

that is f maps each real number 'a' into the set of rational numbers less than 'a'.

f is one-one

Let $a, b \in \mathbb{R}$ s.t.

$a \neq b.$

Suppose, $a < b$, there is a rational number 'r' such that

$a < r < b.$

$\Rightarrow \quad r \in f(b)$ but $r \notin f(a)$

$\Rightarrow \quad f(a) \neq f(b)$

Thus f is one-to-one ~~and onto~~ ~~$P(Q)$~~

$\Rightarrow \quad |\mathbb{R}| \le |P(Q)| = |P(N)|$

$$c \le 2^{\aleph_0} \quad \text{— (1)}$$

Conversely, we need to show that

$$2^{\aleph_0} \le c$$

For this purpose, let $C(N)$ be the collection of all characteristics functions ~~of the~~

$$f : N \longrightarrow \{0, 1\}$$

Also, let $I = [0, 1]$ and consider the function $F : C(N) \to I$.

$$F(f) = 0.\underbrace{f(1)f(2)f(3)\ldots}_{\text{on off}}$$

(N written above $f(1)f(2)f(3)\ldots$)

$A = \{2, 4, 6\}$.

$\chi_A(2) = 1$

$\chi_A(1) = 0$.

اگر point میں کوئی بھی ان میں ایک on اور دوسرا off ہو

Let $f \neq g$, $f, g \in C(N)$

$$0.f(1)f(2)\ldots f(n)\ldots \neq 0.g(1)g(2)\ldots g(n)\ldots$$

$$F(f) \neq F(g)$$

H

f^N

$|P(Q)| =$

Since

$\Rightarrow$

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Hence F is one-to-one function from $C(N)$ to I.

$$|P(Q)| = |P(N)| = |C(N)| \le |I| \quad \text{— (2)}$$

[illegible] solve Q [illegible]

Since $C(N) \sim P(N)$

$$\Rightarrow \quad |C(N)| = |P(N)| = 2^{\aleph_0}. \quad \text{— (3)}$$

(2) and (3)

$$\Rightarrow \quad 2^{\aleph_0} \le c \quad \text{— (4)}$$

From (1) and (4); we have

$$2^{\aleph_0} = c.$$

$$\aleph_0 < c = 2^{\aleph_0}.$$

$$\aleph_0 < 2^{\aleph_0} = c$$

$$\aleph_0 < c$$

Ex. $S = (0,1)$

Let S denotes the open interval $(0,1)$, and let S^{ω} be the set of all denumerable sequences of the form

$X = (x_1, x_2, \ldots, x_n, \ldots)$, where $x_i \in S = (0,1)$.

Then

$$|S^{\omega}| = |S| = c.$$

& Cartesian product [illegible] 27

$A' = A \times \{1\} = \{(a,1), (b,1), (c,1)\}$.

$B' = B \times \{2\} = \{(b,2), (c,2), (f,2)\}$

Since $A \sim A'$, $B \sim B'$.
So $\alpha = |A'|$ and $\beta = |B'|$.
and $A' \cap B' = \varphi$.
$\Rightarrow \alpha + \beta = |A' \cup B'|$.

We can always make a disjoint family of sets.

6.

~~Theorem.~~

Def. Multiplication of Cardinal no.
Let α and β be the cardinal numbers of sets A and B resp

$$\alpha\beta = |A \times B|.$$

Theorem.
Show that addition and Multiplication of cardinal no. satisfy the following properties.

2013
i) ✓ $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

ii) $\alpha + \beta = \beta + \alpha$.

$(\alpha\beta)\gamma = \alpha(\beta\gamma)$

$\alpha\beta = \beta\alpha$

2013

$\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$.

If $\alpha \leq \beta$ then $\alpha+\gamma \leq \beta+\gamma$.

If $\alpha \leq \beta$ then $\alpha\gamma \leq \beta\gamma$.

Proof.

Without loss of generality let A, B and C be mutually (pairwise) disjoint set. s.t $\alpha = |A|$, $\beta = |B|$ and $\gamma = |C|$.

i) L.H.S.

$(\alpha+\beta)+\gamma = |(A \cup B) \cup C|$

$= |A \cup (B \cup C)|$

$= \alpha + (\beta+\gamma)$

ii) L.H.S

$\alpha+\beta = |A \cup B|$

$= |B \cup A|$

$= \beta+\alpha$

$|A \times B||C|$ 29

~~L.H.S~~

iii) $(\alpha\beta)\gamma = |(A \times B) \times C|$

$= |A \times (B \times C)|$

$= \alpha(\beta\gamma)$

iv) $\alpha\beta = \beta\alpha$

L.H.S $= \alpha\beta$

$= |A \times B|$

$A \times B \sim B \times A$

$f(a,b) = (b,a)$

Since $A \times B \sim B \times A$.

$\Rightarrow |A \times B| = |B \times A|$

$\Rightarrow \alpha\beta = \beta\alpha$.

v). If $\alpha \leq \beta$ then $\alpha + \gamma \leq \beta + \gamma$.

Since $\alpha \leq \beta$, there is an injective mapping $f: A \longrightarrow B$

Define $g: A \cup C \longrightarrow B \cup C$ by

$$g(x) = \begin{cases} x, & x \in C \\ f(x), & x \in A \end{cases}$$

g is an injective function from $A \cup C$ into $B \cup C$

$\Rightarrow |A \cup C| \leq |B \cup C|$

$\Rightarrow \alpha + \gamma \leq \beta + \gamma$.

If $\alpha \le \beta$ then $\alpha\gamma \le \beta\gamma$.

Since $\alpha \le \beta$, then there is an injective mapping f i.e

$$f : A \longrightarrow B.$$

Define
$$g : A \times C \longrightarrow B \times C \quad \text{by}$$

$$g(x,c) = \begin{cases} (x, c), & x \in C \\ (f(x), c), & x \in A \end{cases}$$

g is an injective mapping from $A \times C$ to $B \times C$.

i.e $|A \times C| \le |B \times C|$

$\alpha\gamma \le \beta\gamma$.



Example.

① Let n be a finite cardinal number then

$$n + \aleph_0 = \aleph_0$$

Consider $A = \{1, 2, 3, \ldots, n\} \Rightarrow |A| = n$.

$B = \{n+1, n+2, \ldots, n+k, \ldots\} \sim \mathbb{N}$.

$|B| = \aleph_0$

~~Define~~ because

$f : N \rightarrow B$ defined by

$f(i) = n+i \quad ; i = 1, 2, \ldots$

that's why B is denumerable.

$n + \aleph_0 = |A \cup B|$

$= |\{1, 2, \ldots n\} \cup \{n+1, \ldots n+k, \ldots\}|$

$= |\{1, 2, \ldots n, n+1, \ldots\}|$

$= |N|$

$n + \aleph_0 = \aleph_0$

Ex:

Show that Union of two denumerable set is denumerable and hence deduce that $\aleph_0 + \aleph_0 = \aleph_0$

(2) 2016 $\aleph_0 + \aleph_0 = \aleph_0$.

Let $E = \{2, 4, \ldots\}$

$\Rightarrow |E| = \aleph_0$

and

$O = \{1, 3, 5, \ldots\}$

$|O| = \aleph_0$.

Now,

$\aleph_0 + \aleph_0 = |E \cup O|$

$= |\{2, 4, \ldots\} \cup \{1, 3, 5, \ldots\}|$

$= |\{1, 2, 3, \ldots\}|$

$= |N|$

$= \aleph_0$.

$\aleph_0(\aleph_0) = \aleph_0 + \cdots \aleph_0 + \cdots$ (countable times) $= \aleph_0$

$\aleph_0(c) \quad \aleph_0 + \cdots \aleph_0 + \cdots$ (uncountable times) $= c.$

Show that

(3) $c + c = c.$

Since

$[0, 1] = [0, \frac{1}{2}) \cup [\frac{1}{2}, 1]$

$|[0, 1]| = |[0, \frac{1}{2}) \cup [\frac{1}{2}, 1]|$

$c = c + c.$

2012

(4) $\aleph_0 \aleph_0 = \aleph_0.$

Since

$N \times N \sim N$

$\Rightarrow |N \times N| = |N|$

$\aleph_0 \aleph_0 = \aleph_0$

(5) $cc = c.$

Since ($S^2 \sim S$)

$(0, 1) \times (0, 1) \sim (0, 1)$

$\Rightarrow |(0, 1) \times (0, 1)| = |(0, 1)|$

$\Rightarrow cc = c$

ordering of set S.

The set S with partial ordering R (S,R) is called an ordered set or partially ordered set.

Notation.

For partial order R, we use symbol "$\preceq$" notation

general terminology of less than or equal to.

$a \preceq b$

(a preceeds b).

or (b succeed a)

Example.

① The most familiar order relation, called usual order, is $\leq$ (less than or equal to) on N or any subset of R.

2.A/2017

② Let P(S) be the power set of S. Then P(S) is an ordered (poset) w.r.t. the relation of the set inclusion $(P(S), \subseteq)$

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Subset relation

(i) For $A \in P(A)$, $A \subseteq A$.

(ii) If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iii) If $A \subset B$, $B \subseteq C$ then $A \subset C$
for any $A, B, C \in P(S)$

③ The relation of divisibility "|" on N is a partial ordering

i) Ref. $a|a$, $\forall a \in N$.

ii) Anti-Sy. If $a|b$ and $b|a$ then $a = b$. $\forall a, b \in N$

iii) Transitive If $a|b$ and $b|c$ then $a|c$ $\forall a, b, c \in N$.

④ The relation of divisibility "|" on Z is not a partially ordered

Hasse Diagram.

Hasse diagram of a partially ordered sets S is a graphical representation of ordering S.

1) The elements of set S are represented by vertices. (Point)

2) Directed line from a to b represents $a << b$

3) $x < y$ iff there is a sequence of edges (directed line) from vertex 'x' to vertex 'y'.

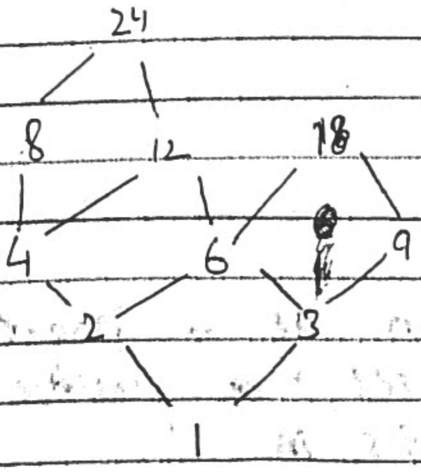
directed graph.
cyclic graph

Example. A/2012.

Let $A = \{2, 3, 4, 6, 8, 9, 12, 18, 24\}$ ordered by divisibility relation i.e,

$$a \leq b \text{ if } a|b.$$

Find minimal, maximal, first and last element of A.



2 and 3 are not comparable because $2 \nmid 3$ and $3 \nmid 2$.

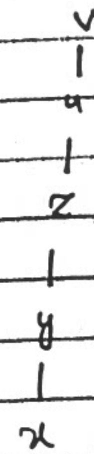
this means any number has more than one immediate predecessor

Remark.

In an ordered set, immediate predecessor of an element is not unique.

Example.

Consider the linearly ordered set given by

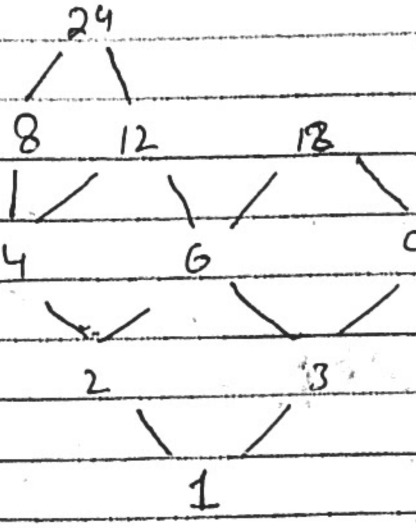


; $S = \{x, y, z, u, v\}$.

Example. A/2012.

Find first and last element of

$S = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ ordered by divisibility.



First element is 1.
Last element does not exist.

Remark.
An ordered set can have atmost one and last element which must be a minimal and

maximal element, respective.

Every first element is minimal but converse may not.

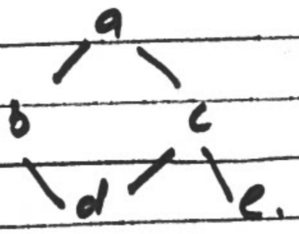
Every last element is maximal but converse may not.

2- S may have neither first nor last element even if S is finite.

3- Finite linearly ordered always has first & last element.

Example

$S = \{a, b, c, d, e\}$.



13th Sep, 2017.

Upper Bound. [lower]

Let S be a (partially) ordered set and let A be a subset of S. An element M of S is called an upper [lower] bound of A if M succeeds [precedes] every element of A,

i.e., $x < M$, $\forall x \in A$.

Supremum [infimum.]

If an upper [lower.] bound precedes [succeeds] all upper [lower] bound of A, then it is called supremum [infimum] of A, denoted by $\sup(A)$.

$\inf(A)$

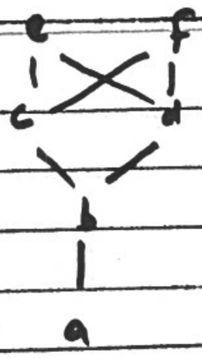
Remark.

An ordered set has atmost one supremum and one infimum. Sometimes, $\sup(A)$ or $\inf(A)$ may not exist.

A/2016, ~~2015~~

Example ① Find U.B, L.B, $\sup(A)$, $\inf(A)$, $S = \{a, b, c, d, e, f\}$.

$A = \{b, c, d\}$ ordered show



U.B = e, f → not comparable

Since e || f sup(A) does not exist.

L.B = a and b.

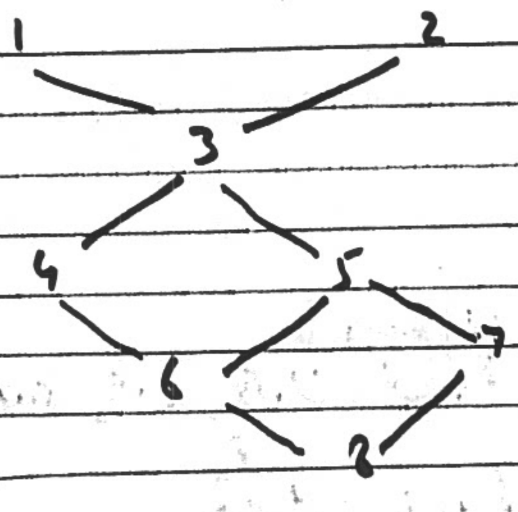
Since a ≤ b.

b = inf(A).

Exp (2)

S = {1, 2, 3, ..., 8}

and A = {4, 5, 7}

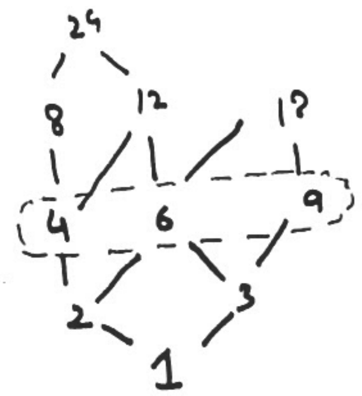


U.B. = 1, 2, 3

Sup(A) = 3.

A/2017.

$X = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$. be ordered as



Let $A = \{4, 6, 9\}$.

i) Find Maximal and minimal element of X.

ii) Find sup and inf of A.

Sol.

i) Maximal element.

An element 'a' of a POset is said to be maximal element if no element in POset strictly succeeds 'a'.

i.e., $x \geq a \Rightarrow x = a$.

~~Isto~~ maximal element in X. are 18, 24.

Minimal element.

An element 'b' of a POset is said to be minimal if no element in POset strictly precedes 'b'

i.e, $x \leq b \Rightarrow x = b$.

Minimal element in X is 1.

Supremum of a set.

If an upper bound precedes all upper bounds of A, then it is called supremum of A, denoted by Sup(A).

Upper bounds of A.

8, 12, 18, 24.

Sup(A)

Sup(A) does not exist because 24 ∥ 18.

Infimum of a set.

If a lower bound succeeds all lower bounds of A, then it is called infimum of A, denoted by inf(A).

lower bounds of A.

1, 2, 3.

Inf(A)

Inf(A) does not exists because 2∥3.

A/2016, A/2014

Q. Consider the set Q of rational numbers with the usual order and consider the subset

$$D = \{x \in Q : 8 < x^3 < 15\}$$

Determine D is bounded above or below?

Sol.

Consider the set

$$D = \{x \in Q : 8 < x^3 < 15\}$$

$$D = \{x \in Q : 2 < x < \sqrt[3]{15}\}.$$

i) D has infinite no. of Upper bounds and lower bound.

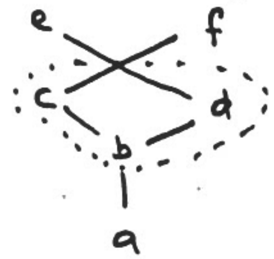
ii) Sup(D) does not exist. because $\sqrt[3]{15} \notin Q$.

iii) Inf(D) is 2

Hence the set D is bounded below.

Qno3(a) A/2016

Let $X = \{a, b, c, d, e, f\}$ be ordered, as



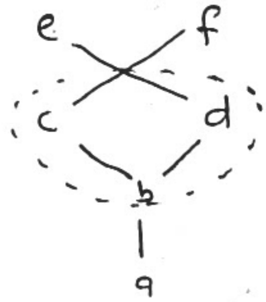
Let $A = \{b, c, d\}$ be a subset of X.

Find minimal, maximal, sup(A), Inf(A).

Sol.

Let $X = \{a, b, c, d, e, f\}$

and $A = \{b, c, d\}$.



- **Minimal element.**

An element of X is said to minimal if no element of X strictly precedes it.

Minimal element of X is 'a'.

- **Maximal element.**

An element of X is said to be maximal if no element strictly succeed it.

Here e and f are maximal elements

- **Upper Bounds.**

An element of X is said to be upper bound if it succeeds every element of A.

Here e, f are upper bounds

Sup(A) sup(A) does not exist because e||f.

- **Lower Bounds.**

An element of X is said to be lower bound if it precedes every element of A.

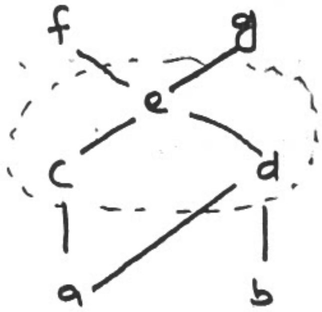
Here lower bounds are a and b

Inf(A).
The lower bound that suceed every lower bound

Inf(A) = b.

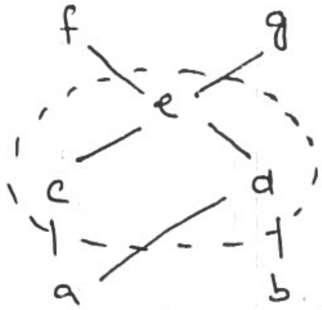
A/2014

Q Let S = {a, b, c, d, e, f, g} be ordered as



and let X = {c, d, e} ⊆ S. Find minimal, maximal, sup(X)
inf(X)

Sol. Let S = {a, b, c, d, e, f, g}
and X = {c, d, e}



· Maximal element.
An element of S is said to be maximal if no element strictly succeeds it.
Here f and g are maximal elements.

· Minimal element.
An element of S is said to be minimal if no element of S strictly precedes it.
Here a and b are minimal elements.

· Sup(X).
The upper bound of S that precedes every upper bound.
Here upper bounds are f and g.
Thus, sup(X) does not exist because f ∥ g.

· Inf(X).
The lower bound of S that succeed every lower bound
Here lower bounds ~~are~~ is 'a'

Hence Inf(X) = a.

A/2015 (State and prove principle of transfinite induction)

A/2017

Statement.

Let S be subset of well-ordered set A with the following properties

$s(a_0) = \phi \subseteq S \Rightarrow a_0 \in S$.

i) $a_0 \in S$, where a_0 is the first element of A.

ii) If $p(a) \subseteq S$, then $a \in S$.

Then $S = A$.

If $A = N$

i) $1 \in S$,

ii) $\{1, 2, 3, \ldots n-1\} \subseteq S$ then $n \in S$

↑ $p(n)$

then $S = N$.

Proof.

On contrary suppose $A \neq S$. It implies that

$A \setminus S \neq \phi$.

Let

$T = A \setminus S \subseteq A$.

Since A is well-ordered set, therefore T has first element t_0

Consider initial segment, $s(t_0)$

Then each element $x \in s(t_0)$ strictly precedes t_0.

$\Rightarrow x \notin T = A \setminus S$

$\Rightarrow x \in S$.

$\Rightarrow \quad S(t_0) \subseteq S.$

Then by condition (ii)

we have

$$t_0 \in S.$$

A contradiction to the fact that $t_0 \notin S$.

Thus our supposition that

$$A \setminus S \neq \phi$$ is wrong.

Hence our claim $S = A$ is true.

Remark.

Note that the condition (i) is a consequence of condition (2), Since

$$S(a_0) = \phi \subseteq S \Rightarrow a_0 \in S.$$

Example.

Consider the set Q of rational numbers with usual ordering.

Note that Q is linearly ordered.

But no element in Q has an immediate predecessor and immediate successor.

For if $a, b \in Q$ s.t.
$a < b$, then we have
$c = \frac{a+b}{2}$ such that

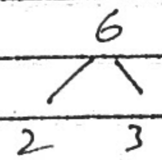
$a < c < b$.

The above example shows that linearly ordered set need not has immediate succesor ~~and~~ or predecessor.

However, this is not true in case of well-ordered sets.

A/2013
A/2014 Another diff b/w L.O and W.O.

Theorem. Every element in a well-ordered set has a unique immediate successor except the last element.



have not unique

$2 << 6$
$3 << 6$

Proof. Let $a \in A$ and $N(a)$ be the ~~set~~ set of element of A which strictly succeed a.
i.e; $N(a) = \{x \in A : a < x\} \subseteq A$.

~~$s(1) = \{\}$~~

Since A is well-ordered set and
$M(a) \subseteq A$.
therefore M(a) has a first element, say b.
We claim that b is the immediate successor of 'a'.

For if, b is not immediate ~~pre~~ successor, then there is some c, s.t. $a < c < b$. It implies that $c \in M(a)$, a contradiction to the fact that b is the first element of M(a).
Hence b is the immediate successor.

Uniqueness.
We show that b is the unique immediate successor of a.
~~For~~ Contrary suppose that 'd' is also immediate successor of 'a'.
$\Rightarrow d \in M(a)$.
Since b is "the" first element of M(a).
So, $b < d$.
or $a < b < d$.
a contradiction that d is immediate successor of a.

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Initial segment

$\overset{i}{S}(4) = \{1, 3, 5, \dots, 2\}$

Exc 8.2.

$A \simeq S(A)$

$f: A \to S(A)$

$f(a) = S(a)$ $\quad S(a) \subseteq S.$

Similarity between Well-ordered sets and its subsets.

Relation b/w image and preimage.

$N \cong E$ $\quad \{1, 2, 3, \dots\}$

$f: N \to E$ $\quad \{2, 4, 6, \dots\}$

$f(x) =$ $\quad x \leq f(x)$

Theorem. A|2018

Let A be a well-ordered set and B be a subset of A. Also,

$$f: A \to B$$

be a similarity mapping of A ont B. Then for each $a \in A$

$$a \leq f(a).$$

Proof. Let A be a well-ordered set and $B \subseteq A$. Also ~~f: A→~~ B be a similarity mapping. contradictory set

Let $D = \{x \mid x \in A \text{ s.t } f(x) < x\}$

If $D = \phi$. Then nothing to prove.

If $D \neq \phi$

Since $D \subseteq A$ and A is a well-ordered set, therefore d_0 be the first element of D.

As $d_0 \in D$, it implies that

$$f(d_0) < d_0 \quad \text{--- (1)}$$

Since f is a similarity mapping therefore if $a < b$

$f(a) < f(b)$ order preserving.

Set $a = f(d_0)$ and

$$b = d_0$$

Then

$$f(f(d_0)) < f(d_0).$$

$f(d_0) \in D$, a contradiction to the fact that d_0 is the first element.

Thus for each $a \in A$, $a \leq f(a)$.

A/2012, A/2011
A/2013. We can define only one mapping for isomorphism between two well ordered sets

Theorem. Let A and B be the similar well-ordered sets. Then there exists only one similarity mapping of A onto B.

Let A and B be two well ordered similar set.

Proof. On contrary, suppose that,

let $f: A \to B$ and

$g: A \to B$ be two distinct isomorphisms

$\Rightarrow f \neq g$.

i.e, there exist, $x \in A$ s.t

$f(x) \neq g(x)$

Since B is the well ordered set and hence linearly ordered, therefore either

$f(x) < g(x)$

or

$g(x) < f(x)$

let

$f(x) < g(x)$.

Since $g: A \to B$ is an isomorphism then $g^{-1}: B \to A$ is also an isomorphism.

It implies that $g^{-1} of$ and $g^{-1} o g$ is also isomorphisms from A to A.

i.e.,

$g^{-1} o f: A \xrightarrow{f \; B \; g^{-1}} A$

and

$g^{-1} o g: A \to A$ are isomorphism.

Since ~~f is an isomorphism~~, therefore

$$f(x) < g(x)$$

$$g^{-1}(f(x)) < g^{-1}(g(x))$$

$$(g^{-1}\circ f)(x) < (\underbrace{g^{-1}\circ g}_{I(x)})(x)$$

$$(g^{-1}\circ f)(x) < x.$$

image precedes its preimage

A contradiction to the fact that

$x \leq g^{-1}\circ f(x)$ (last theorem).

Thus the supposition that there are two isomorphisms ~~leads~~ to the contradiction mentioned above.

A/2012.

Theorem.

A well-ordered set can not be similar to one of its initial segment.

Proof.

Let A be a well-ordered set.

On contrary, we suppose that ~~let~~ A is similar to its initial segment $s(a)$ for some $a \in A$.

There is an order-isomorphism

$f: A \to S(a)$

It implies that for $a \in A$. $f(a) = x < a$, $x \in S(a)$

$f(a) < a$,

a contradiction to the fact that $a \le f(a)$.

$N = \{1, 2, 3, \ldots\}$. $S(2) = 1$.

$\{1, 3, 5, \ldots, 2, 4, 6, \ldots\}$.

$S(2) \notin N$.

i.e. $\{1, 3, 5 \ldots\} \notin N$.

Theorem.

① Two different initial segments of a well-ordered set can not be similar.

② Let A and B be well ordered sets and let an initial segment $S(a)$ of A be similar to an initial segment of B. Then $S(a)$ is similar to unique initial segment of B.

Since $S(a')$ ~~has unity~~ is similar to unique initial segment of $S(b)$, therefore

$$f(S(a')) = S(b')$$

$$\Rightarrow S(a') \cong S(b'). \quad \text{from } ①$$

Theorem. A/2016.

Let A be B be a well-ordered set and S be subset of A with the following property

if $a \leq b$ and $b \in S$ then $a \in S$.

Then $S = A$ or S is an initial segment of A.

Proof.

Let A and B be a well-ordered set and S be subset of A

If $S = A$. Then nothing to prove.
Let $S \neq A$. Then define a set

$$T = ~~S \setminus A~~ \; A \setminus S.$$

Since $T \subseteq A$, a well ordered set, therefore T has a first element, say a.

Suppose x be an arbitrary element of $s(a)$

i.e $x \in s(a)$

$\Rightarrow x < a$.

$\Rightarrow x \notin T$

$\Rightarrow x \in S$

$\Rightarrow s(a) \subseteq S$. — (1)

Let

$b \in S$ and $b \notin s(a)$ (contrary)

$\Rightarrow b \nless a$

$\Rightarrow a \leq b$ }

But $b \in S$. }

by given condition

$a \in S$, but $a \in T = A \setminus S$

a contradiction to the fact that $a \notin S$.

Hence our supposition, $b \notin s(a)$ is false

$b \in s(a)$

$\Rightarrow S \subseteq s(a)$ — (2)

From (1) and (2)

$S = s(a)$.

Theorem Let A and B be well ordered set and let

$$S = \{ x : x \in A ;\ s(x) \simeq s(y),\ \text{where } y \in B \}.$$

then $S = A$ or S is an initial segment of A.

Proof. Let $x \in S$ and s.t. $x' < x$ ($x' \in A$)

Since $x \in S$, therefore

$s(x) \simeq s(y)$ for some $y \in B$.

By previous result, we know that

$s(x') \simeq s(y')$. for $y' \in B$. s.t $y' < y$

$\Rightarrow$ $x' \in S$.

$x' \leq x$ and $x \in S$, $x' \in S$.
($a \leq b$, $b \in S$, $a \in S$)

$\Rightarrow$ $S = A$ or S is an initial segment of A.

Comparison of Well-ordered Sets.

(interms of order)

Suppose A and B are well-ordered sets and A is similar to initial segment of B, then A is said to be shorter than B and B is said to be larger than A.

$A \cong s(b)$. $b \in B$.

or

$B \cong s(a)$ $a \in B$.

imp A/2013 A/2014

Theorem.

Let A and B be well-ordered set and also let

$$S = \{x : x \in A ;\ s(x) \cong s(y), \text{where } y \in B\}$$ (S ⊆ A)

and

$$T = \{y : y \in B ;\ s(y) \cong s(x) ; \text{where } x \in A\}.$$ (T ⊆ B)

Then $S \cong T$.

Proof.

Let $x \in S$, then $s(x) \cong s(y)$ for some $y \in B$.

We know that for each $s(x)$, $s(y)$ is unique and

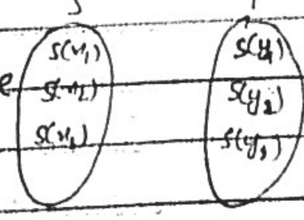
Q and $x' < x < f(x)$! $\quad x' < f(x')$
$x < f(x)$?

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Sets.

ll-ordered
ial
to be
d to

vice versa.
Let
$f: S \to T$ defined by
$y = f(x)$ if $S(x) \simeq S(y)$.



Obviously, f is one-to-one & onto
We need to show that
f preserves the order.

Let $x, x' \in S$ and $y = f(x)$, $y' = f(x')$
and
$x' < x$.
$\quad x' < f(x')$
$\quad x < f(x)$
We show $f(x') < f(x)$.
$\Rightarrow$ Since $x' \in S$, $S(x') \simeq S(f(x')) = S(y')$

well-

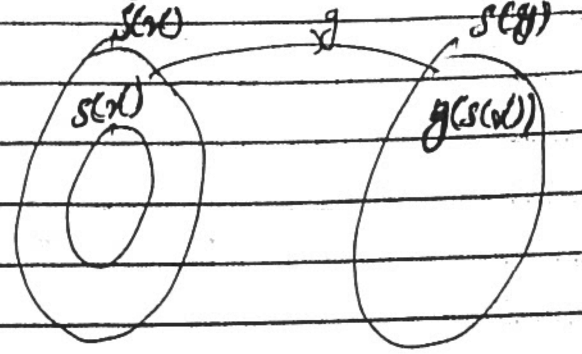
Since $S(x) \simeq S(y)$, there exist an isomorphism
$g: S(x) \to S(y)$.

$\subseteq B$
where $y \in B\}$

Given that

$\subseteq B$
where $x \in A\}$.

$x' < x$
$g(x') < g(x)$ $\because g$ is isomorphism.



S(y)

P(x).

It implies that restriction of g on $S(x') \subseteq S(x)$ is also an isomorphism.

$$S(x') \simeq S(g(x'))$$

Since $g(x') \in S(y)$, therefore

$$g(x') < y.$$

Since $S(x')$ can be similar to a unique initial segment of B.

$$S(g(x')) = S(f(x')).$$

$$\Rightarrow \quad g(x') = f(x')$$

or

$$f(x') = g(x') < y = f(x)$$

g is defined on initial segment but f is defined on S and T. f and g are not same.

$$\Rightarrow \quad f(x') < f(x)$$

$$\Rightarrow \quad y' < y.$$



a first element, say $s(a)$. That is

$$s(a) \leq s(x)$$

Ordinal Numbers. A/2017

Consider a collection F of well-ordered sets.

Each well-ordered set A in F is assigned the symbol such that, any two well-ordered sets are assigned the same symbols if A and B are similar, i.e., order-isomorphic.

This symbol is called Ordinal number of A, denoted by.

$$\lambda = \text{ord}(A).$$



Remark.

① Since isomorphism (order-iso), is an equivalence relation in any collection of ordered sets.

Thus all well-ordered sets are partitioned into disjoint classes of isomorphic / Similar sets.

Hence ordinal number can be viewed as an equivalence class of all similar well-ordered sets,

$$C_\lambda \cap C_\mu = \phi.$$

Class of all

Read it.

Structure of ordinals

Theorem: Any set of ordinal numbers is linearly ordered by the relation $\lambda \leq \mu$.

Proof.

Let λ and μ be the ordinal numbers of well-ordered sets A and B. Then we know that any of the following three possibility can arise

1. $A \simeq B$
2. $A \simeq S(b) \subseteq B$.
3. $B \simeq S(a) \subseteq A$.

From (1)
$\Rightarrow \lambda = \mu$

From (2)
$\Rightarrow \lambda < \mu$

From (3)
$\Rightarrow \lambda > \mu$.

Since A and B was arbitrary. Hence it is true for all well-ordered sets in general.

Exc.

Let $\lambda = \mathrm{ord}(A)$, where A is a well-ordered set. Also $\mu < \lambda$. Then there is unique initial segment $s(a)$ of A s.t

$$\mu = \mathrm{ord}(s(a)).$$

Proof.

Let $\mu = \mathrm{ord}(B)$, where B is some well-ordered set.

Since $\mu < \lambda$, therefore B is shorter than A

$\Rightarrow B \simeq s(a) \subseteq A$. where $s(a)$ is an initial segment of A.

$\Rightarrow \mathrm{ord}(B) = \mathrm{ord}(s(a))$

$\Rightarrow \mu = \mathrm{ord}(s(a))$.

For uniqueness

Let $\mu = \mathrm{ord}(s(b))$, $a \neq b$,

a contradiction to the fact that $s(a) \simeq s(b)$

Because no two initial segments are unique in a well-ordered set.

A/2011

Theorem. (Set of ordinal numbers is well-ordered)

Let $s(\lambda)$ be the set of ordinal numbers less than the ordinal λ

Then $\lambda = \mathrm{ord}(s(\lambda))$.

Proof.
Let A be a well ordered set such that
$$\lambda = ord(A)$$
Also let $S(A)$ be the collection of all the initial segments of A.
Then we know that,
$$A \simeq S(A) \qquad f(a) = S(a)$$
$$\Rightarrow \lambda = ord(S(A)).$$
We need to show that $S(\lambda) \simeq S(A)$
For this purpose let $\mu \in S(\lambda)$, then there is a unique initial segment $S(a)$ such that $\mu = ord(S(a))$.
Then we can define a function
$$f : S(\lambda) \longrightarrow S(A), \text{ as}$$

μ کسی بھی Initial segmt
کو اسی وقت map کریں گا
جب اس کی ordinal value ہو گی۔

$$f(\mu) = S(a) \quad \text{if } \mu = ord(S(a)).$$

Then the function is **one-to-one**.
Let $\mu \neq \nu$
$$\Rightarrow S(a) \neq S(b). \qquad \text{s.t } \mu = ord(S(a)),\ \nu = ord(S(b)).$$
$$f(\mu) \neq f(\nu).$$

f is onto.
Let $S(b) \in S(A)$
Then $S(b) \subseteq A$
$\Rightarrow S(b)$ is shorter than A.
$\Rightarrow ord(S(b)) < ord(A)$
$ord(S(b)) < \lambda$

f is

let ~~ν~~ $\nu = ord(S(b))$.

$\Rightarrow \quad \nu < \lambda$

$\Rightarrow \quad \nu \in S(\lambda)$

~~Thus~~ Hence for every $S(b) \in S(A)$ $\exists$ $\nu \in S(\lambda)$ such that

$$\nu = ord(S(b))$$

$$f(\nu) = S(b).$$

f is Order preserving.

let $\mu, \nu \in S(\lambda)$ and $\mu < \nu$,

we are to show that

$$f(\mu) < f(\nu)$$

Since $\mu < \nu$.

~~ord~~ $\Rightarrow S(a) \subseteq S(b)$ $\left\{ \begin{array}{l} \text{s.t } \mu = ord(S(a)) \\ \nu = ord(S(b)) \end{array} \right.$

$\Rightarrow f(\mu) < f(\nu)$

Hence

$$S(\lambda) \simeq S(A)$$

$$ord(S(\lambda)) = ord(S(A))$$

$$ord(S(\lambda)) = \lambda.$$

A/2012.
A/2017

Theorem: Let λ be any ordinal number. Then $\lambda+1$ is the immediate ~~predecessor~~ successor of λ.

Proof

Let μ be imediate successor of λ. Then by def

$\mu = \text{ord}(S(\lambda) \cup \{\lambda\})$

$= \text{ord}(S(\lambda)) + \text{ord}(\{\lambda\})$

$\mu = \lambda + 1$ $\quad \because \lambda = \text{ord}(S(\lambda))$

$\Rightarrow \lambda+1$ is immediate successor of λ.

For non-disjoint set.

$C = \{A_i : i \in I\}$ where A_i's are not disjoint.

Let $\text{ord}(A_1) = \lambda_1 = \text{ord}(A_1 \times \{1\}) = \text{ord}(B_1)$

$\text{ord}(A_2) = \lambda_2 = \text{ord}(A_2 \times \{2\}) = \text{ord}(B_2)$

$\vdots$

$\text{ord}(A_i) = \lambda_i = \text{ord}(A_i \times \{i\}) = \text{ord}(B_i)$

$\lambda_1 + \lambda_2 + \cdots + \lambda_i = \text{ord}(B_1; B_2; \ldots; B_i)$

$= \text{ord}(B_1) + \text{ord}(B_2) + \cdots + \text{ord}(B_i)$

$\lambda = ord(\{1, 2, \textcircled{3} \underline{4}\})$

 $ord(S(\lambda)) = 3.$ $S(\lambda) \cup \{4\}$

From the ~~general addition of~~ ordinal numbers

$1 + 1 + 1 + \dots = \omega.$

$A_1 = \{1\}$, $A_2 = \{2\}$, $\vdots$, $A_n = \{n\}$, $\vdots$ $\Rightarrow$

$ord(A_i) = 1 \quad \forall\, i = 1, 2, \dots$

$ord(\{A_1, \dots, A_n, \dots\}) = 1 + 1 + 1 + \dots$

$= ord(\{1, 2, 3, \dots\})$

$= ord(N) = \omega.$

nal

diate

sor

$s > ord(S(A))$

or of

are

$. \, ord(B_1)$

$ord(B_2)$

$ord(B_i)$

$i.)$

$\dots + ord(B_i)$

consists of exactly one element from each A_i

$$B \subseteq \bigcup_{i \in I} A_i$$

and

$$B \cap A_i \neq \Phi$$

$$B \cap A_i = \{a_i\} \quad \text{for all } i \text{ ?}$$

Theorem. A/2016, A/2017.

Show that Axiom of choice is equivalent to Zermelo's postulate.

Proof.

Let $C = \{A_i : i \in I\}$ be a non-empty collection of non-empty ~~set~~ disjoint sets.
By Axiom of choice, there exist a choice function f on C.
Define a set $B = \{f(A_i) : i \in I\}$.
Then

$$B \subseteq \bigcup_{i \in I} A_i$$

$$B \cap A_i = \{f(A_i)\} \quad \forall i \in I$$

Conversely,
Now, let $C = \{A_i : i \in I\}$ be a non-empty collection of non-empty sets which may or not be disjoint.
Define

$$A_i^* = A_i \times \{i\} \text{ for each } i \in I.$$

Then

$$C^* = \{A_i^* : i \in I\}$$

is a nonempty family of non-empty disjoint sets.

By Zermelo's postulate, there is a subset $B \subseteq \bigcup_{i \in I} A_i^*$ such that

$B \cap A_i^*$ ~~and~~ contains exactly one element of A_i^*.

Let

$$B \cap A_i^* = \{(a_i, i)\} \quad \forall i \in I$$

Let us define a function

~~$f : B \cap A_i^*$~~

$$f : \{A_i\} \to \bigcup_{i \in I} A_i \quad \text{by}$$

$$f(A_i) = a_i \quad \text{if} \quad \{(a_i, i)\} \in B \cap A_i^*, \quad a_i \in A_i$$

Zorn's lemma. (A/2016 Def.)

Let X be a partially ordered set in which every chain has an upper bound, then there exist a maximal chain (element) in X.

Well-ordering Theorem. A/2015

Every non-empty set X can be well ordered.

Proof. (On the base of Zorn's lemma we want to prove W.O theorem.)

Let X be a nonempty set and $\mathcal{A}$ be the family of all well-ordered subsets of X.

$$\mathcal{A} = \{ W \mid W = (B, \leq_B) \}$$ → ordering → set inclusion

We can define a partial order $\preceq$ on $\mathcal{A}$ as

If W_1 and $W_2 \in \mathcal{A}$, then ($\preceq$, $\leq$ diff)

$$W_1 \preceq W_2$$

$W_1 = (B_1, \leq_{B_1})$

$W_2 = (B_2, \leq_{B_2})$

If either $B_1 \subseteq B_2$ or

[illegible] set [illegible] set [illegible] ordering [illegible]

combination proven

if $\begin{cases} B_1 \subseteq B_2 \\ B_1 \text{ has same ordering as that of } B_2 \end{cases}$ [illegible] set [illegible] set [illegible]

Now define a collection $\mathcal{A}$

chain

$\{ W_i ; i \in I \} \subseteq \mathcal{A}$ a linearly

ordered family of subsets of $\mathcal{A}$ (ordered by ~~Continuation~~)

It implies that $\{B_i : i \in I\}$ is also a linearly ordered collection w.r.t set inclusion (first component of W_i)

$\Rightarrow$ Let $B = \bigcup_{i \in I} B_i$ and ~~a~~

$W = (B, \leq_B)$ ~~(be the)~~

If $a, b \in B$, then there are i and j such that

$a \in B_i$, $b \in B_j$, $i \neq j$

Since $\{B_i\}$ is linearly ordered by set inclusion

$B_i \subseteq B_j$ or $B_j \subseteq B_i$

$\Rightarrow a, b \in B_i$ (say) $j < i$ i.e. $B_j \subset B_i$

Since B_i is a well ordered and hence a linearly ordered set, then

(say) $a <_{B_i} b$ as elements of B_i

$\Rightarrow a <_B b$

It implies that W is a well-ordered set and thus

$W \in \mathcal{A}$

Since $B_i \subseteq \bigcup_{i \in I} B_i$, $\forall i$

Hence W is an upper bound of $\{W_i\}$.

and

ordered

α

$(\alpha, \leq)$

(B_1, $\leq_{B_1}$)

(B_2, $\leq_{B_2}$)

$\mathcal{A}$

early

ordering

Sound

~~Since~~ $\mathcal{A}$ is POset and $\{W_i\}$ is chain has an upper ...

Then by Zorn's lemma, $\mathcal{A}$ has a maximal element, say W^*.

We need to show that $W^* = X$

$\leftarrow$ wo ہے ... ہے [illegible]

On contrary, suppose that $W^* \neq X$

Then there is y such that

$$y \in X \backslash W^*$$

$\hat{W} = \{W^*; y\}$, Then $\hat{W}$ is also well ordered set and hence

$$\hat{W} \in \mathcal{A}$$

~~Since~~ Note that W^* is an initial segment of $\hat{W}$ and therefore

$W^* \subset \hat{W}$

A contradiction to the fact that W^* is a maximal element.

$\Rightarrow$ $W^* \neq X$ our assumption is wrong

Hence $W^* = X$

$\mathcal{A} \rightarrow$ define partial order.

$B \rightarrow$ U.B., $\rightarrow \hat{W} \rightarrow$ Maximal Element.

Zorn lemma

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Application of Zorn's Lemma to Algebra.

A/2017.

Ex. Let R be a ring with unity 1. Then every proper ideal J is contained in maximal ideal.

Recall that.

(1) Proper ideal. An ideal is a proper ideal of ring R if $J \subset R$.

(2) An ideal J of R(ring) with identity 1 is proper ideal iff $1 \notin J$.

(3) Maximal ideal. An ideal M is maximal if no other ideal K properly lies between M and R.
i.e., $M \notin K \subset R$
or $M = K$ or $M \subset R$.

Proof.

Let J be a proper ideal of ring R and let $\mathcal{A}$ be a collection of all proper ideals of R which contains J

$$\mathcal{A} = \{ M_\alpha \subset R,\ M_\alpha \text{ is proper ideal},\ J \subset M_\alpha \}$$

~~Let C~~

Note that the set $\mathcal{A}$ is a partially ordered set w.r.t. set inclusion($\subseteq$)

Let us define $C = \{M_i\} \subset \mathcal{A}$ be a collection of linearly ordered sets

$\le$ / or $\le$ ... UB

Then by def of linear ordering either

$$M_i \subset M_j \text{ or } M_j \subset M_i$$

Let $M_i < M_j$ $i<j$

Define $M = \bigcup_i M_i$ $1 \notin M_i$ for all i

M, being union of ascending chain of ideal is also an ideal. (nested)

Furthermore, since $1 \in M_i$ $\forall i \in I$, as M_i's are proper ideals.

Therefore $1 \in \bigcup_i M_i$.

Hence M is a proper ideal containing J

$\Rightarrow M \in \mathcal{A}$

Also; M; being union of all proper ideals is an upper bound of chain $C = \{M_i\}$.

Then by Zorn's lemma $\mathcal{A}$ contains a maximal ideal, say M^*, which contains J

⟵⟶

Exc. Prove that every non-empty vector space has a basis.

maximal set of L.I

Proof.

Let V be a vector space.

Case I. V consists of zero vector alone, 0.

Then $\{\}$ is a basis for 0

Case II.

Let V contains at least one non-zero vector, say a.

Let

$\mathcal{A}$ be a collection of independent set of vectors in V.

$$\mathcal{A} = \{ B : B \subseteq V, \text{ all vectors in } B \text{ are L.I} \}$$

∴ Then $\mathcal{A}$ (poset) is non-empty since $\{a\} \in \mathcal{A}$

Now define

$C = \{B_i\}$ be a linearly

2nd Oct, 2017

A/2014, A/2013, A/2011

Theorem. Let X be a partially ordered set. Then X contains a maximal chain i.e., (A chain | linearly ordered set which is not a proper subset of any other chain).

(A/2011) Prove that every finite partially ordered set has at least one minimal and one maximal element.

Proof.

The result holds for empty set and finite sets.

Let X be a non-empty set (P.O set) $(X, \leq_p)$ $(N, |)$
i.e $X \neq \phi$

Using well-ordering theorem, X can be well-ordered with $\to (X, \leq_w)$. $(N, \leq)$

جب initial seg. اور first element کی بات کریں گے تو '$\leq_w$' order use کریں گے۔

first element x_0.

Let well-ordering be denoted by $\leq_w$ and partial ordering be denoted by $\leq_p$.
Hence the terms like initial segment and first element will now be used w.r.t $\leq_w$

The term "comparable" will be used w.r.t partial $\leq_p$.

جس order کو استعمال کرتے ہوئے ہم نے $\leq_p$ کے تحت ایک چین نکالی ہے
وہ maximal ہوگی

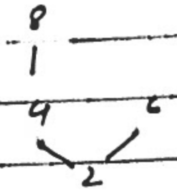
Let A be an initial segment of X
(A can be X)

A function $f: A \to A$ is called a special function if s.t

$$f(x) = \begin{cases} x & \text{if } x \text{ is comparable to each element in } f[S(x)] \\ x_0 & \text{otherwise} \end{cases}$$

(comparable w.r.t. $\leq_p$; $S(x)$ w.r.t. $\leq_w$)

Recursive def



$A = \{4, 2\}$
$f(4) = 1$
$S(4) = \{2\}$,
$f(S(4)) = \{f(2)\} = ?$

Then we show that the special function f is unique if it exists.

On contrary suppose f and f' be two special functions on A.
Let a_0 be the first (w.r.t. w.o) of such element for which $f \neq f'$
i.e., $f(a_0) \neq f'(a_0)$.

Then f and f' agree on $S(a_0) = \{x \mid x < a_0\}$, $x \in A$.

Case I.
If a_0 is not comparable to any element of $f[S(a_0)] = f'[S(a_0)]$
then $f(a_0) = x_0 = f'(a_0)$.

Case II.

Otherwise $f(a_0) = a_0 = f'(a_0)$

$\Rightarrow \quad f(a_0) = f'(a_0)$, a contradiction to the fact that

$$f(a_0) \neq f'(a_0)$$

Hence f is unique.

Next; let $\mathcal{I} = \{A_i\}$ be a collection of all those initial segments for which a special function, say f_i, exists.

Let us define $B = \bigcup_i A_i$ — (1)

Then, B, being union of initial segment is also an initial segment.

Also, B admits a special function

$$g : B \to B$$

defined by (also special)

$$g(b) = f_i(b), \text{ if } b \in A_i$$

(f_i can be viewed as a restriction of g on A_i)

Next, we show that $B = X$.

On contrary suppose that

$$X \setminus B \neq \phi$$

Then let $y \in X \setminus B$.

Then $C = B \cup \{y\}$ is also an initial

τ τ

segment.

Moreover C admits a special function $h: C \to C$ defined by

$$h(c) = \begin{cases} g(C) & ; \text{ if } C \in B \\ y & ; \text{ if } y \text{ is comparable with each initial segment of } h(B). \\ x_0 & ; \text{ otherwise} \end{cases}$$

Hence, C, being initial segment with special function $h(x)$, $C \in \tau$

$\Rightarrow C \subseteq \bigcup A_i = B$.

$\Rightarrow B \cup \{y\} \subseteq B \Rightarrow y \in B$, a contradiction

Hence $B = X$.

$g(B) = g(X)$

$a \in X$, $a \in g(X)$

3rd Oct, 2017.

Next, we claim that $g[B]$ i.e., $g[X]$ is a maximal chain.

Suppose on contrary, ~~$g[X]$~~ let S be another chain s.t.

$g[X] \subset S \subseteq X$. — ①

Let z be the first element in $S \setminus g[X]$

Note that

$$\boxed{g[X] = S(z)}$$ — ②

$S = g[X] \cup \{z, z_1, z_2 \ldots z_n\}$

Since S is a chain, z is comparable ($\leq_p$) to every (each) element of S, In particular z is comparable to each element of $g[X]$ ($g[X] \subseteq S$.)

$S(z) \subseteq X$

$g[S(z)] \subset g[X]$. — ③

~~Also $z \in S \Rightarrow$~~

(3) $\Rightarrow$ z is comparable to each element of $g[S(z)]$. $\Rightarrow g(z) = z$.

Since $z \in X$ and $g(z) = z \in g[X]$

A contradiction to the supposition that $z \notin g[X]$

Hence (~~our~~ claim ~~that~~) $g[X]$ is a maximal chain.

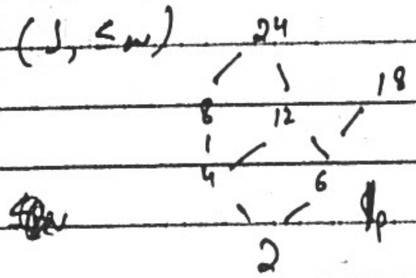
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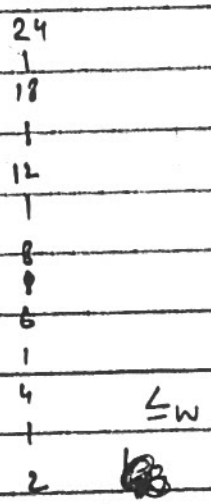
Assignment

$S = \{2, 4, 6, 8, 12, 18, 24\}$.

$(S, \leq_p)$

$(S, \leq_w)$





Find $g[S] = ?$

Ex.

There exist a finite set of natural numbers which is not a proper subset of any other finite set of natural numbers.

Proof.

Let $\mathcal{A}$ be the collection of all finite sets of $\mathbb{N}$ partially ordered

By set inclusion, $\subseteq$.
Let $C = \{M_i\}$ be a chain in $\mathcal{A}$.

Define M_i not finite.

$M = \bigcup M_i$ (countable union)

Then each $M_i \subseteq M$. Hence M is the upper bound of C. But $M \notin A$. Thus by Zorn's lemma, $\mathcal{A}$ has a maximal element (a finite set) not properly contained in any other element.

Sec II 83

Pre-requisite / Preliminaries.

Algebra of Sets.

A collection $\mathcal{A}$ of subsets of X is called an algebra if for any $A, B \in \mathcal{A}$

i) $A \cup B \in \mathcal{A}$ $\Rightarrow \phi \in \mathcal{A}$

ii) $A^c \in \mathcal{A}$ $\Rightarrow X \in \mathcal{A}$

Ring.

i) $A \cup B \in R$ $\Rightarrow \phi \in R$

ii) $A \cap B^c \in R$ $\Rightarrow X \notin R$.

A/2013 Qno5(a)

Theorem. Every Algebra is a ring but converse may not be true.

Example $X = N$

$R = \{A \subseteq N : A \text{ is finite}\}$.

Let $A, B \in R$.

then A is finite and B is finite.

So that

i) $A \cup B$ is finite $\Rightarrow A \cup B \in R$.

ii) $A \cap B^c$ is finite $\Rightarrow A \cap B^c \in R$.

Then R is a ring.

Since A^c is infinite ; $A^c \notin R$.

R is not an algebra.

A/2013, A/2015 Qno 5(a)

Prove that $\mathcal{A} = \{ A \subseteq \mathbb{N} : A$ is finite or A^c is finite $\}$ is algebra of sets but not σ-algebra.

<u>σ-Algera.</u>

An algebra $\mathcal{A}$ is said to be a σ-algebra if for any sequence of set $\{A_n\}$ in $\mathcal{A}$

$$\bigcup_n A_n \in \mathcal{A}$$

let $A, B \in \mathcal{A}$

then A is finite or A^c is finite.

and B is finite or B^c is finite.

(i) Algebra of sets.

i) $A \cup B$ is finite $\Rightarrow$ $A \cup B \in \mathcal{A}$.

or $A^c \cup B^c$ is finite $\Rightarrow$ $A^c \cup B^c \in \mathcal{A}$.

ii) A^c is finite $\Rightarrow$ $A^c \in \mathcal{A}$

or $(A^c)^c = A$ is finite $\Rightarrow$ $A \in \mathcal{A}$

Hence $\mathcal{A}$ is algebre of sets

(ii) σ- Algebra.

Consider a sequence of of sets $\{A_n\}$ or $\{A_n^c\}$ that are finite in $\mathcal{A}$

Now $\bigcup_n A_n$ is not finite.

or

$\bigcup_n A_n^c$ is not finite

because countable union of finite sets is countable.

Hence $\bigcup_n A_n \notin \mathcal{A}$

or

$\bigcup_n A_n^c \notin \mathcal{A}$.

Thus $\mathcal{A}$ is not σ-algebra.

Remark. Any σ-algebra is closed under countable intersection.

Let $F = \{A_1, A_2, \ldots\}$.

Since $A_n^c \in \mathcal{A}$ $\forall n$. ($\mathcal{A}$ is Algebra)

By def of σ-algebra

$$\bigcup_n A_n^c \in \mathcal{A}.$$

Since $\mathcal{A}$ is σ-algebra.

$$\left(\bigcap_n A_n\right)^c \in \mathcal{A}$$

$$\Rightarrow \bigcap_n A_n \in \mathcal{A}$$

Thus countable intersection also belongs to σ-algebra

Example. Intersection of any class of σ-algebra is also a σ-algebra.

Let $F = \{\mathcal{A}_i : i \in I\}$

We show $C = \bigcap_i \mathcal{A}_i$ is also a σ-algebra.

let $A, B \in C$
$\Rightarrow A, B \in \mathcal{A}_i \quad \forall i \in I.$
Since each $\mathcal{A}_i$ is σ-algebra
Therefore,
i) $A \cup B \in \mathcal{A}_i \quad \forall i \in I$
$\Rightarrow A \cup B \in \bigcap_i \mathcal{A}_i$
Let ii) $A^c \in C.$
$\Rightarrow A^c \in \mathcal{A}_i \quad \forall i \in I.$
$\Rightarrow A^c \in \bigcap_i \mathcal{A}_i \quad \forall i \in I$

Hence $C = \bigcap_i \mathcal{A}_i$ is also σ-algebra.

σ- Ring.
A ring R is said to be σ-Ring if for any sequence of sets in R,
$$\bigcup A_n \in R.$$

Remark. A/2011 Qno5(a)
A ring may not be σ-Ring.
Let $X = N$.
$R = \{A \subset N : A \text{ is finite}\}$

$\Rightarrow \bigcup_i A_i$ is infinite $\forall A_i \in R$
$\Rightarrow \bigcup_i A_i \notin R$

Thus R is not σ-Ring.

Length ko generalized krna h (Measure theory)
[illegible] apply Reiman [illegible] openset [illegible]
based Reiman

Reiman based on interval not on (open) set
because we not able to define length on this set

Finite topology $= \tau = \{A_1, \dots, A_n\} =$ σ-Algebra

i) $\phi, X \in \tau$, $\phi, X \in \mathcal{A}$.
ii) $\bigcup_{i=1}^{n} A_i \in \tau$, $\bigcup_{i=1} A_i \in \mathcal{A}$.
iii) $\bigcap_{i=1}^{n} A_i \in \tau$, $\bigcap_{i=1} A_i \in \mathcal{A}$

if

Remark.

The union of two σ-algebras may not be sigma algebra.

e.g., $X = \{a, b, c, d\}$.

$\mathcal{A}_1 = \{\phi, X, \{a\}, \{b, c, d\}\}$ σ-algebra

$\mathcal{A}_2 = \{\phi, X, \{b\}, \{a, c, d\}\}$ σ-algebra

$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 = \{\phi, X, \{a\}, \{b\}, \{a, c, d\}, \{b, c, d$

$\mathcal{A}$

Theorem. A/2016. Q no 5(a).

Given any collection C of subsets of a set X, there is a smallest σ-algebra containing C.

Proof.
Let F be the collection $\{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$ of all algebras containing C.
Note that $F \neq \phi$, since $P(X) \in F$ and $C \subseteq P(X)$.
Now, let us define

$$\mathcal{A} = \bigcap_i \mathcal{A}_i$$

Since $C \subseteq \mathcal{A}_i$ for each $i \in \mathbb{N}$, therefore

$$C \subseteq \bigcap_i \mathcal{A}_i$$
$$\Rightarrow C \subseteq \mathcal{A}$$

Further $\mathcal{A}$, being intersection of algebras $\mathcal{A}_i$, is also an algebra.

We need to show that $\mathcal{A}$ is the smallest algebra containing C.
On contrary suppose $\mathcal{B}$ is an algebra containing C such that

$$\mathcal{B} \subseteq \mathcal{A} \quad \text{--- (1)}$$
$$\Rightarrow \mathcal{B} \in F$$

$\mathcal{A}$, being the intersection of all

algebras containing $\mathcal{C}$,

$$\mathcal{A} \subseteq \mathcal{B} \rightarrow (2)$$

Our assumption is wrong. Hence $\mathcal{A}$ is smallest algebra containing $\mathcal{C}$.

Proposition. Let $\mathcal{A}$ be a σ-algebra of sets. If $\{A_n\}$ is any sequence of sets in σ-algebra $\mathcal{A}$ then there exist $\{B_n\}$, a sequence of pairwise disjoint sets in σ-algebra $\mathcal{A}$ s.t.

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

- [illegible] overlap [illegible]

Proof. Let us define B_i's as follow.

$$B_1 = A_1$$

$$B_2 = A_2 \setminus A_1$$

$$B_3 = A_3 \setminus (A_1 \cup A_2)$$

$$\vdots$$

$$B_n = A_n \setminus (A_1 \cup A_2, \ldots \cup A_{n-1}).$$

$$\vdots$$

$(X, \mathcal{A}, \mu)$ is called measure space

Example. A/2015 Q no 5 (a).

let $X = N$

$\mathcal{A} = P(N) = 2^N$

$\mu : \mathcal{A} \to \overline{\mathbb{R}}$

counting measure.

$$\mu(A) = \begin{cases} \text{no. of points in } A & ; A \text{ is finite} \\ \infty & ; A \text{ is infinite.} \end{cases}$$

$\mu([0,1]) = \infty$

Is μ a measure? but $l([0,1]) = 1$ and $1 \neq \infty$.

$\Rightarrow m(I) \neq l(I)$

Sol.

i) $\mu(\phi) = 0$.

ii) $\mu(A) \geq 0$

iii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ A_i's are pairwise disjoint.

Case I

Let all A_i's are finite except A_j

$\Rightarrow \bigcup_i A_i$ is infinite.

by definition of μ, $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty$ — ①

Also

$$\sum_{i=1}^{\infty} \mu(A_i) = \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \mu(A_i) + \mu(A_j)$$

$= \infty$ — ②

Countable additive.

Case II. If All A_i's are finite
then
$$\bigcup_{i=1}^{\infty} A_i \text{ is infinite}$$
$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty.$$

Also
$$\sum_{i=1}^{\infty} \mu(A_i) = \infty$$

Thus countably Additive.

Thus μ is a measure.

Example

Let $X = N$

$$\mu(A_i) = \begin{cases} 0 & A_i \text{ is finite} \\ \infty & A \text{ is infinite} \end{cases}$$

① $\mu(\phi) = 0.$

② $\mu(A) \geq 0.$

③ Countably Additive. , A_i's are pair wise disjoint.

Case I

let A_i's are finite except A_j

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty$$

$$\sum_{i=1}^{\infty} \mu(A_i) = \infty$$

finite

infinite.

).

are

-se disjoint.

A_i

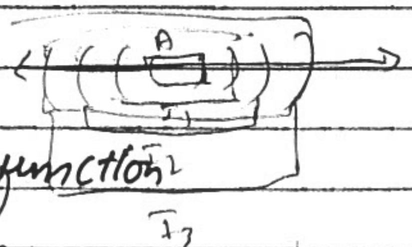
$= \infty$. ①

Then $\bigcup_i A_i \neq \phi$

$\Rightarrow \mu^\circ(\bigcup_i A_i) = 1$

Also $\sum_{i=1}^{\infty} \mu(A_i) =$

imp A/2017 Q no 5(a).



Example. 1st step toward target.

Consider the set function

$$\mu^*: 2^{\mathbb{R}} \longrightarrow [0,\infty].$$

defined by (we interested in decreasing length)

sum of lengths of open covers.

$$\mu^*(A) = \inf\left\{\sum_n l(I_n) : A \subseteq \bigcup_n I_n\right\}.$$

where $\{I_n\}$ is a countable open cover of A. Show that μ^* is outer measure.

Solution

Note that

i) $\mu^*(\phi) = 0$

Because we know that for any $\varepsilon > 0 \in$

$$\phi \subseteq \left(\frac{-\varepsilon}{4}, \frac{\varepsilon}{4}\right)$$

$$\mu^*(\phi) \leq l\left\{\left(\frac{-\varepsilon}{4}, \frac{\varepsilon}{4}\right)\right\}$$

↓
Infimum

$$= \frac{\varepsilon}{4} - \left(-\frac{\varepsilon}{4}\right)$$

$$= \frac{\varepsilon}{2} < \varepsilon.$$

$\therefore \mu^*(\phi) < \varepsilon$ — (1)

Since $\varepsilon > 0$ is arbitrary, the inequality (1) holds only if
$\mu^*(\phi) = 0$.

ii) Monotone property.
if $A \subseteq B$
we show $\mu^*(A) \leq \mu^*(B)$
let $A \subseteq B$.

$\mu^*(A) = \inf \{ \sum_n l(I_n) : A \subseteq \bigcup_n I_n \}$

$< \inf \{ \sum_n l(J_n) : B \subseteq \bigcup_n J_n \}$

as $\bigcup_n I_n \subseteq \bigcup_k J_k$

$\Rightarrow \mu^*(A) \leq \mu^*(B)$.

13th Oct, 2017

iii) Countably subadditive. $\mu^*(\bigcup A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Let $C = \{A_n\}$ be a countable collection of subsets of $\mathbb{R}$. $(C \subseteq 2^{\mathbb{R}})$

We have to prove that

$$\mu^*(\bigcup_n A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n) \quad (1)$$

$\mu^*([1,\infty))$ unbounded

Case I (Trivial case)

If any of sets $\{A_n\}$ is unbounded

Then $\bigcup_n A_n$ is also unbounded.
Further, $\mu^*(\bigcup A_k) = \infty$, if A_k
if A_k is unbounded for some $k \ge 1$
In this case the relation (1) holds trivially.

Case II (Non-trivial case)

Let $\mu^*(A_n) < \infty$ for all $n \in N$
all bounded

Let $\varepsilon > 0$, then for each n, by definition of $\mu^*(A_n)$, there is a countable collection ~~$\mu^*(A_n)$~~ of open intervals $\{I_{n,i}\}$ s.t

$$A_n \subseteq \bigcup I_{n,i} \quad (2) \quad n \text{ fix}$$

By applying μ^*, i varies.

$$\sum_i \ell(I_{n,i}) < \mu^*(A_n) + \varepsilon/2^n \quad (3)$$

$\downarrow$ inf

inf $s \le x$ then we always find $\varepsilon > 0$, $s + \varepsilon > x$.

Since (2) $\Rightarrow$ $A_n \subseteq \bigcup_i I_{n,i}$

$\mu^*(A_n) \le \mu^*(\bigcup_i I_{n,i})$

$\mu^*(A_n) \le \sum_i \ell(I_{n,i})$

$\downarrow$ inf

$\mu^*(A_n) + \frac{\varepsilon}{2^n} > \sum_i \ell(I_{n,i})$

$$\bigcup_n A_n \subseteq \bigcup_n \left(\bigcup_i I_{n,i}\right) \quad — (4)$$

Note that $\bigcup_n \bigcup_i I_{n,i}$ being countable union of countable sets $\left(\bigcup_i I_{n,i}\right)$ is countable and contains $\bigcup_n A_n$.

$$\Rightarrow \mu^*\left(\bigcup_n A_n\right) \le \sum_n \sum_i l(I_{n,i}) \quad — (5)$$

inf

From eq (5) and (3)

$$\mu^*\left(\bigcup_n A_n\right) < \sum_n \left[\mu^*(A_n) + \frac{\varepsilon}{2^n}\right]$$

$$\mu^*\left(\bigcup_n A_n\right) < \left[\sum_n \mu^*(A_n) + \sum_n \varepsilon/2^n\right]$$

$$= \sum_n \mu^*(A_n) + \varepsilon(1) \qquad S_\infty = \frac{1/2}{1-1/2}$$

$$\Rightarrow \mu^*\left(\bigcup_n A_n\right) < \sum_n \mu^*(A_n) + \varepsilon \quad \forall \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, therefore

$$\mu^*\left(\bigcup_n A_n\right) \le \sum_n \mu^*(A_n).$$

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definition,
ollection

inf

$\le x$
n we always
find $\varepsilon > 0$
$\varepsilon > x$.

$\le \mu^*\left(\bigcup_i I_{n,i}\right)$

$\le \sum_i l(I_{n,i})$

$\le > \sum_i l(I_{n,i})$

Lebesgue Outer Measure.

- [illegible] in fact

Def. یہ def [illegible] length [illegible] interval measure [illegible] objective [illegible]

Lebesgue Outer measure is a set function $m^*: 2^{\mathbb{R}} \to [0, \infty]$.

$$m^*(A) = \inf\left\{\sum_n l(I_n) : A \subseteq \bigcup_n I_n\right\}.$$

where infimum is taken over countable sequence of open intervals containing or covering A, and l denotes the length of the interval I_n.

Remark.

We have seen several examples of outer measures. However, Lebesgue outer measure is an attempt which generalizes the notion of length to any subset A of $\mathbb{R}$.

ie, $m^*(I) = l(I)$

i) Def. of length of interval

ii) Translation invariant

iii) Countably additive [illegible]

[illegible] removal [illegible] subset [illegible] $2^{\mathbb{R}}$ [illegible]

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Q no 5(b) A/2016, A/2012 Q no 5(a)

Theorem. Lesbesgue outer measure m^* of an interval is its length, i.e.,

$$m^*(I) = l(I).$$

Proof.

Case I.

Let I be a closed and bounded interval. That is

$I = [a, b]$ for some $a, b \in \mathbb{R}$

We have to show that

$$m^*([a,b]) = m^*(I) = b-a \qquad \geq, \leq$$

We know that for each $\varepsilon > 0$

$$[a,b] \subset (a-\varepsilon, b+\varepsilon)$$

By def of infimum.

$$m^*(I) = m^*([a,b]) \leq l[(a-\varepsilon, b+\varepsilon)]$$

$$= b-a+2\varepsilon$$

Since $\varepsilon > 0$ is an arbitrary real no., therefore.

$$m^*(I) \leq b-a. \quad \text{---} \ (1)$$

Next, we show that
$m^*(I) \geq b-a$.

This is equivalent to showing that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of open ~~but~~ bounded intervals covering ~~A~~, I,

$$\sum_{k=1}^{\infty} l(I_k) \geq b-a$$

Since $[a,b]$ is closed and bounded, therefore by Hein-Borel Theorem, $I=[a,b]$ is compact. That is, any collection $\{I_m\}_{m=1}^{\infty}$ of open intervals covering I, has a finite subcover $\{I_j\}_{j=1}^{n}$.

The proof will be complete if we show that
$\sum_{m=1}^{\infty} l(I_m) \geq \sum_{j=1}^{n} l(I_j) \geq b-a$

Since $a \in \bigcup_{i=1}^{n} I_i$, there is some I_k in this collection $\{I_j\}_{j=1}^{n}$ such that $a \in I_k$. Denote I_k by (a_1, b_1) i.e.,

$$a_1 < a < b_1$$

If $b \in (a_1, b_1)$, then the proof is complete.
$b_1 > b$ and $a_1 < a$

$\Rightarrow$
$\Rightarrow \sum$
If
i.e.
If
If not
from
We
until
Since
interv
Thus
subc
Now $b_n \geq$
Now
Thus
m^*

$\Rightarrow b_1 - a_1 > b - a$ $\qquad a_1 < a < b < b_1$

$\Rightarrow \sum_{k=1}^{n} l(I_k) > l(I_k) > b - a.$

hat

lection

ring

If $b_1 \notin (a_1, b_1)$, then set $b \in (a_2, b_2)$

i.e., $a_2 < b < b_2$

If $b \in (a_2, b_2)$, the proof is complete

If not, then select another interval (a_3, b_3) from $\{I_j\}_{j=1}^{n}$ s.t. $b_2 \in (a_3, b_3)$

We continue this selection process until it terminates.

Since these are only finite no. of intervals, I_k.

Thus, using this process, we obtain a subcollection $\{(a_j, b_j)\}_{j=1}^{N}$, $N \leq n$, s.t.

d,

$[a, b]$

$\}_{m=1}^{\infty}$

Now $b_N > b$ and $a_1 < a \Rightarrow b_N - a_1 > b - a \longrightarrow (\circledast)$

$a_{k+1} < b_k \Rightarrow a_{k+1} - b_k < 0$

$b_N < b$

now

Now
$$\sum_{i=1}^{n} l(I_i) \geq \sum_{j=1}^{N} l(a_j, b_j)$$

$$= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \cdots + (b_2 - a_2) + (b_1 - a_1)$$

$$= b_N - (a_N - b_{N-1}) - (a_{N-1} - b_{N-2}) - \cdots - (a_2 - b_1) - a_1$$

$$> b_N - a_1 \quad \because (\text{since } a_{k+1} - b_k < 0)$$

I_k,

$$> b - a \quad \text{using } (\circledast)$$

Thus
$$m^*(I) = \inf\left\{\sum_n l(I_n) : I \subseteq \bigcup_n I_n\right\} \geq b - a.$$

$\checkmark$

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This proves that

$m^*([a,b]) = b-a$

Case II. If I is an open and bounded interval (e.g., $I = (a,b)$) infinity not included

~~Then~~ کیونکہ case I پورا ہو جاتا ہے

Let $\bar{J}$ be a closed interval such that

$x \in A$. $\bar{J} \subset \bar{I}$. Then for any $\varepsilon > 0$ we can write $m^*(\bar{J}) \le l(I)$.

$$m^*(\bar{J}) + \varepsilon > l(I)$$

or $$l(I) \le m^*(\bar{J}) \quad \text{——} (1)$$

inf $= m^*$ x A

we always find $\varepsilon > 0$ s.t. $m^* + \varepsilon > x$

$m^*(\bar{J}) = \inf\{\sum l(J_n) : J \subset \bigcup_n I_n\}$

$\downarrow$ $\subset \mathbb{R}$

$\Rightarrow$ m^* (inf) $\in \mathbb{R}$

- for $x \in A$, we find ε s.t. $m^* + \varepsilon > x$

(ii) Study completness property.

- for $\varepsilon > 0$ we can find $x \in A$ s.t. $m^* + \varepsilon > x$.

Since m^* is monotonic set function therefore

$m^*(\bar{J}) \le m^*(I) \quad \text{——} (2)$

From inequality (1) and (2) we have

$$l(I) < m^*(I) + \varepsilon$$

Since $\varepsilon > 0$, is an arbitrary real no., therefore

$$l(I) < m^*(I) \quad \text{—— (3)}$$

Also, $I \subset \bar{I}$ (→ closed bounded)

$$\Rightarrow m^*(I) < m^*(\bar{I}) \qquad \because \text{monotonicity}$$
$$= l(\bar{I}) \qquad \text{using case I}$$
$$= l(I) \qquad \because l((a,b)) = b-a$$
$$\Rightarrow m^*(I) < l(I) \quad \text{—— (4)} \qquad l([a,b]) = b-a$$

From (3) and (4)

$$m^*(I) = l(I).$$

Case III. When I is an open and unbounded interval.

Then for any closed set

$$m^*(J) \leq m^*(I). \qquad J \subset I.$$

Let $m^*(J) = \alpha$

Then, $m^*(I) > a$, $\forall \alpha \in \mathbb{R}$

Then $m^*(I) = \infty = l(I)$

A/2011 Q no 6(a) ✓

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N, N×N, Q

Ex. If $A \subseteq \mathbb{R}$ is countable then $m^*(A) = 0$.

Proof.

Case I.

When A is a singleton set

i.e

$A = \{x\}$

For any $\varepsilon > 0$ we can define

$I = (x - \varepsilon/2, x + \varepsilon/2)$

$$m^*(A) \leq m^*(I) \quad A \subset I.$$
$$= l(I)$$
$$= x + \varepsilon/2 - x + \varepsilon/2$$
$$= \varepsilon$$

$m^*(A) \leq \varepsilon$.

$\Rightarrow m^*(A) = 0$.

Because $\varepsilon > 0$ is arbitrary.

Case II. If A is denumerable

we can write A as a sequence of infinite seq distinct terms.

$A = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$

or $A = \bigcup_{n=1}^{\infty} \{x_n\}$

$m^*(A) = m^*(\bigcup \{x_n\})$

So

Sinc

Ex.

Ou

Proof.

I =

W

f

$m^*(A) = 0.$

Since m^* is countably subadditive

So

$$\leq \sum_i m^*(\{x_n\}) = 0+0+\dots$$

$$\Rightarrow m^*(A) \leq 0. \quad \text{—(1)}$$

Since m^* is non-negative, therefore

$$m^*(A) \geq 0 \quad \text{—(2)}$$

From (1) and (2)

$$m^*(A) = 0.$$

Ex. $[0,1]$ is uncountable using Lebesgue outer measure.

Proof.

On contrary, suppose that $I = [0,1]$ is countable

$$\Rightarrow m^*(I) = 0. \quad \text{—(1)}$$

We know that

$$m^*(I) = l(I) = 1. \quad (2)$$

from (1) and (2), contradiction

Hence

$[0,1]$ is uncountable

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V.imp A/2017

Translation Invariant.

$$m^*(A) = m^*(A+y)$$

(1) $m^*(A) = \inf\{\sum_n l(I_n) : A \subset \bigcup_n I_n\}$.

(2) If $A \subseteq \bigcup_n I_n$

then $A+y \subseteq \bigcup_n (I_n+y)$

$A+y \subseteq \bigcup_n (J)$ — length of interval does not change by translation.

$\therefore l(I_n) = l(I_n+y)$

$\Rightarrow m^*(A+y) = \inf\{\sum_n l(J_n) = \sum l(I_n) : A \subseteq \bigcup_n J_n\}$.

$\Rightarrow m^*(A) = m^*(A+y)$

26th Oct, 2017.

A/2017 Q no 6 (b)(ii)

Exc. If $m^*(A) = 0$, then

$$m^*(A \cup B) = m^*(B).$$

Proof

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

$$m^*(A \cup B) \leq m^*(B) \quad (1)$$

Since

$B \subseteq A \cup B$

by monotonicity

$x + y \in \bigcup_n J_n$

Thus $E + y \subseteq \bigcup_n J_n$.

iii) $l(I_n) = l(J_n)$ implies that

$$\sum_n l(I_n) = \sum_n l(J_n)$$

Therefore,

$$\inf\left\{\sum_n l(I_n) : E \subseteq \bigcup_n I_n\right\} = \inf\left\{\sum_n l(J_n) : E + y \subseteq \bigcup_n J_n\right\}$$

Thus

$$m^*(E) = m^*(E + y).$$

A/2017 Q no 6 (a)
A/2014 Q no 5 (a)

Theorem.

Show that m^* is translation invariant.

Proof.

By the translation of set E of real numbers by any real number y, we mean the set

$$E+y = \{x+y : x \in E\}.$$

We show that

$$m^*(E+y) = m^*(E).$$

Case I : If E is countable.

If E is countable set, then $E+y$ is also countable

So for any countable set

$$m^*(E+y) = 0 = m^*(E).$$

Case II: If E is uncountable.

Suppose E is uncountable.

y def of m^*

$$m^*(E) = \inf\left\{\sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_n I_n\right\}$$ when I_n is a

;uence of open intervals.

Put $I_n + y = J_n = \{x+y : x \in I_n\}$.

lote that

i) J_n is open interval because I_n is open interval for each n.

) If $x+y \in E+y$, then $x \in E \subseteq \bigcup_n I_n$, which gives $x \in I_i$, for some i,

Now $x+y \in I_i + y = J_i$ for some i and so

P.T.O

A/2016, Q no 5 (a)(ii), A/2014 Q no 5(a)(ii)

Proposition.

Given any set A and any $\epsilon > 0$, there is an open set O such that $A \subseteq O$ and $m^*(O) < m^*(A) + \epsilon$.

Proof.

By definition of $m^*(A)$, for $\epsilon > 0$, there exists a countable collection $\{J_n\}$ of open intervals such that

$$A \subseteq \bigcup_n J_n$$

and

$$m^*(A) + \epsilon > \sum l(J_n) \longrightarrow (*)$$

Put $O = \bigcup_n J_n$. Now, O, being union of open intervals, is open

$$m^*(O) = m^*(\bigcup_n J_n) \leqslant \sum_n m^*(J_n) = \sum_n l(J_n)$$

$$m^*(O) < m^*(A) + \epsilon \qquad \text{using } (*)$$

where $A \subseteq O$.

Hence

$$m^*(A \cap E) + m^*(A \cap E^c) = m^*(A)$$

Remark

If E is countable, then $m^*(E) = 0$.

If $m^*(E) = 0$, then E is measurable.

Every countable set is measurable.

21 Oct, 2017

Theorem. A/2017. Q no 6(b)(i)

Let E_1 and E_2 be measurable sets, Then $E_1 \cup E_2$ is also measurable

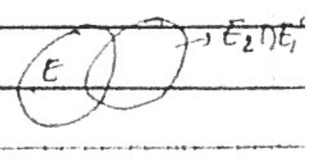
Proof.

We have to prove that for any subset $A \subseteq \mathbb{R}$

$$m^*(A) = m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$

Note that

$$E_1 \cup E_2 = E_1 \cup (E_2 \cap E_1^c)$$



Further

$A\cap(E_1\cup E_2) = A\cap[E_1\cup(E_2\cap E_1^c)]$

$= (A\cap E_1)\cup[A\cap(E_2\cap E_1^c)]$

$= (A\cap E_1)\cup[(A\cap E_1^c)\cap E_2]$.

$m^*(A\cap(E_1\cup E_2)) = m^*[(A\cap E_1)\cup\{(A\cap E_1^c)\cap E_2\}]$

Using subadditive property

$m^*(A\cap(E_1\cup E_2)) \le m^*\{(A\cap E_1)\} + m^*\{(A\cap E_1^c)\cap E_2\}$ —— (1)

Now consider

$m^*[A\cap(E_1\cup E_2)] + m^*[A\cap(E_1\cup E_2)^c]$.

$\le m^*[A\cap E_1] + m^*[(A\cap E_1^c)\cap E_2] + m^*[(A\cap E_2^c\cap E_1^c)]$

Given that E_2 is measurable

$m^*(A\cap(E_1\cup E_2)) + m^*[A\cap(E_1\cup E_2)^c]$

$\le m(A\cap E_1) + m^*(A\cap E_1^c) =$

$= m^*(A)$

Collection of all measurable sets is an algebra.

2 ✓

A/2017 Qno6 b(ii)

Exc. If E_1 and E_2 are measurable, then $E_1 \cap E_2$ is measurable

~~Since $E_1 \cup E_2$~~

Let E_1 and E_2 are measurable
the E_1^c and E_2^c are also measurable
and $E_1^c \cup E_2^c$ is also measurable.
Now
$E_1^c \cup E_2^c = (E_1 \cap E_2)^c$ is mesurable

Hence $E_1 \cap E_2$ is measurable

Theorem. Finite union of measurable sets is measurable.

Proof. (By induction)

Let $\{E_i\}_{i=1}^{n}$ be a finite collection of measurable sets.

For $n=1$,
$\bigcup_{i=1}^{1} E_i = E_1$ is measurable.

Let the statement be true for $n=k-1$.

That is $\bigcup_{i=1}^{k-1} E$ is measurable

We need to prove $\bigcup_{i=1}^{k} E_i$ is measurable.

Consider

$$\bigcup_{i=1}^{k} E_i = \left(\bigcup_{i=1}^{k-1} E_i\right) \cup E_k$$

Then, ~~again~~ being union of two measurable sets is measurable.

Remark.

Similarly, finite intersection of measurable sets is measurable.

Theorem. Let A be any subset of $\mathbb{R}$ and $\{E_1, E_2, \ldots, E_n\}$ be a finite family of pairwise disjoint measurable sets. Then

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Proof. By induction,

For $k=1$

$$m^*\left[A \cap \left(\bigcup_{i=1}^{1} E_i\right)\right] = \sum_{i=1}^{1} m^*(A \cap E_i)$$

3rd Nov, 2017 ✓ 114

Theorem: Let E be any set of real numbers then the following statements are equivalent.

17/2018 [Also do for more]

(i) E is measurable

(ii) For each $\varepsilon > 0$ there is an open set O containing E for which *(softer version)*

$$m^*(O \setminus E) < \varepsilon,$$

Approximation level is my hand

(iii) There is a G_δ set such that G, containing E for which *(more general version)*

$$m^*(G \setminus E) = 0.$$

↑ *Best approximation*

__Proof__

$(i) \Rightarrow (ii)$.

Let E be measurable set.

Case I When $m^*(E) < \infty$

By def of infimum, for every $\varepsilon > 0$ there is countable collection of open intervals $\{I_n\}$ s.t $E \subseteq \bigcup_n I_n$

$$\sum_n \ell(I_n) < m^*(E) + \varepsilon \quad \text{—} (1)$$

Define $O = \bigcup_n I_n$, then O, being

union of open intervals, is an open set containing E

$$m^*(\mathcal{O}) = m^*\left(\bigcup_n I_n\right)$$

Since m^* is countably subadditive

$$m^*(\mathcal{O}) \leq \sum_n m^*(I_n)$$

$$= \sum_n l(I_n)$$

Using ①

$$m^*(\mathcal{O}) < m^*(E) + \varepsilon$$

$$m^*(\mathcal{O}) - m^*(E) < \varepsilon$$

Using the excision property

$$m^*(\mathcal{O} \setminus E) < \varepsilon$$

Case II. When $m^*(E) = \infty$

Then E can be expressed as countable union of disjoint measurable set $\{E_k\}_{k=1}^{\infty}$ each of which has a finite measure.

i.e., $m^*(E_k) < \infty$

Thus $E \subseteq \bigcup_k E_k$ —— ②

Using case I, there is an open set O_k s.t.

$$E_k \subseteq O_k \text{ and } m^*(O_k \setminus E_k) < \varepsilon/2^k \quad -③$$

Define

$$O = \bigcup_k O_k$$

Then $E \subseteq \bigcup_k O_k$

$$O \setminus E = \left(\bigcup_k O_k\right) \setminus E$$

$$= \bigcup_k (O_k \setminus E)$$

$$\subseteq \bigcup (O_k \setminus E_k)$$

$E \subseteq \bigcup_k E_k$

$E \supseteq E_k$

$E^c \subseteq E_k^c$

$O_k \cap E^c \subseteq O_k \cap E_k^c$

$O_k \setminus E \subseteq O_k \setminus E_k$

$$m^*(O \setminus E) \leq m^*\left[\bigcup (O_k \setminus E_k)\right]$$

$$\leq \sum_k m^*(O_k \setminus E_k)$$

$$< \sum_k \varepsilon/2^k \quad \text{from ③}$$

$$= \varepsilon \sum_k \left(\frac{1}{2}\right)^k$$

$$= \varepsilon (1)$$

$$\Rightarrow m^*(O \setminus E) < \varepsilon.$$

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$\Rightarrow x \in (C \setminus B) \cup (A \setminus C)$	$\Rightarrow x \in (C \setminus A) \cup (B \setminus C)$

$$\Rightarrow \quad x \in (C \setminus B) \cup (A \setminus C) \cup (C \setminus A) \cup (B \setminus C)$$

$$x \in (B \Delta C) \cup (A \Delta C)$$

$$x \in (A \Delta C) \cup (B \Delta C)$$

↓ σ - algebra
↓ Set function (Example measure)
↓ outer measure
↓ Lesbegue outer measure
↓ ↓ ← Measurable set (Introduce).
· Lebesgue measure

A/2016 (Q no6 (a))

Theorem: Let E be a subset of $\mathbb{R}$ with $m^*(E) < \infty$. The set E is measurable iff for given $\varepsilon > 0$, there is a finite union, H, of finite open intervals such that

$$m^*(E \Delta H) < \varepsilon.$$

$m^*(E \Delta H)$. $E \Delta H = (E \setminus H) \cup (H \setminus E)$

angular
-arity.

Proof. Given that E is measurable set. Then there is an open set O containing E s.t

$$m^*(O \setminus E) < \varepsilon/2 \quad \text{—(1)}$$

for any given $\varepsilon > 0$.

(By definition of open set, O can be expressed as a union)

By Excision property

$$m^*(O \setminus E) = m^*(O) - m^*(E) < \varepsilon/2$$

$$\Rightarrow \quad m^*(O) < m^*(E) + \varepsilon/2$$

Then $m^*(O)$ is also finite

i.e., $m^*(O) < \infty$ ②

~~Then~~

Also, O being an open set can be expressed as union of pair wise disjoint finite open interval. ~~[x x x x x x]~~

i.e., $O = \bigcup_{i=1}^{\infty} I_i$ $\because$ infinite $\because m^*(O) < \infty$

② $\Rightarrow m^*(O) = m^*\left(\bigcup_{i=1}^{\infty} I_i\right) < \infty$ $\because E$ = finite measurable

Equality

$$\Rightarrow \sum_{i=1}^{\infty} m^*(I_i) < \infty \qquad m^*\left[\mathbb{R} \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right] = \sum_{i=1}^{\infty} m^*(\mathbb{R} \cap E_i)$$

So, the series $\sum_{i=1}^{\infty} m^*(I_i)$ ~~is~~ $m^*(\cup E_i) = \sum m^*(E_i)$

converges in $\mathbb{R}$.

Then there is $n \in \mathbb{N}$ s.t.

$$\left|\sum_{i=n+1}^{\infty} m^*(I_i)\right| < \varepsilon/2 \rightarrow ③ \quad i \geq n$$

Let

$$H = \bigcup_{i=1}^{n} I_i$$

Finite interval means finite length.

Now, Consider

$$E \Delta H = (E \setminus H) \cup (H \setminus E)$$

Since $E \subseteq O$

$$\Rightarrow E \setminus H \subseteq O \setminus H$$

Also

$$H \subseteq O$$

$$\Rightarrow H \setminus E \subseteq O \setminus E$$

Now

$$E \Delta H \subseteq (O \setminus H) \cup (O \setminus E)$$

By monotonicity of m^*

$$m^*(E \Delta H) \leq m^*(O \setminus H) + m^*(O \setminus E)$$

$$\leq m^*\left[\left(\bigcup_{i=1}^{\infty} I_i\right) \setminus \left(\bigcup_{i=1}^{n} I_i\right)\right] + \varepsilon/2$$

$$= m^*\left(\bigcup_{i=n+1}^{\infty} I_i\right) + \varepsilon/2$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon.$$

Converse part use A, B, C result.

Conversely, suppose that there is a finite union H of finite open intervals

i.e.,

$$H = \bigcup_{i=1}^{n} I_i \quad \text{s.t} \quad m^*(E \Delta H) < \varepsilon \quad \rightarrow (1)$$

We need to prove, E is measurable

$\sqrt{\ }$ 120

~~By def of m^* (inf), for any $\varepsilon > 0$~~

$\exists \{J_n\}_{n=1}^{\infty}$ of open intervals such that

$$\sum_{n=1}^{\infty} \ell(J_n) < m^*(E) + \epsilon$$

Set $O = \bigcup_{n=1}^{\infty} J_n$

$$m^*(O) = m^*\left(\bigcup_n J_n\right)$$

$$\leq \sum_n \ell(J_n) \quad \text{subadditive}$$

$$< \underbrace{m^*(E)}_{\text{finite}} + \epsilon \longrightarrow (2)$$

So $m^*(O)$ is also finite.

Next, define $J = O \cap H$ (measurable)

Then $m^*(J) < m^*(O) < \infty$

Also, J is measurable

Then, by Excision property, for any $\varepsilon > 0$

$$m^*(O \setminus J) < \epsilon \longrightarrow (3)$$

since $E \subseteq O$.

$O \setminus E = E \Delta O$

Consider

$O \Delta E \subseteq (O \Delta J) \cup (J \Delta E)$

(~~By monotonicity we have~~) $\longrightarrow$ (4)

Note that

~~$J \setminus E$~~ $=$

$J \subseteq H$

$J \setminus E \subseteq H \setminus E$

$E \setminus J = E \cap (O \cap H)^c$

$= (E \cap O^c) \cup (E \cap H^c)$

$E \setminus J = \phi \cup E \cap H^c = E \setminus H$

$J \triangle E = (J \setminus E) \cup (E \setminus J)$

$J \triangle E \subseteq E \triangle H \quad \rightarrow (5)$

(4) and (5) $\Rightarrow$

$O \triangle E \subseteq (O \triangle J) \cup (E \triangle H)$

By monotonicity.

$m^*(O \triangle E) \leq m^*(O \triangle J) + m^*(E \triangle H)$ — Since $J \subseteq O$, $O \triangle J = O \setminus J$

$\leq m^*(O \setminus J) + \varepsilon$

$< \varepsilon + \varepsilon = \epsilon'$

$\Rightarrow m^*(O \triangle E) < \varepsilon'$

or

$m^*(O \setminus E) < \varepsilon'$ as $O \setminus E \supseteq O \triangle E$

Thus

E is measurable.

$E \subseteq O$.

$\setminus E$

✓ 122^{-2+4}

Example

Find Lebesgue measure of the following sets.

① $A = (-2, 6)$ $\quad m(A) =$

A/2012 Qno5(b)

② $B = [3, 5] \cup [-4, -2]$ $\quad m(B) = 2 + 2 = 4$

③ $C = \bigcup_{k} \left\{ x \in \mathbb{R} : \frac{1}{2^k} \leq x \leq \frac{1}{2^{k-1}} \right\}$

④ $A = Q$ A/2012 Qno5(b) $\quad m(Q) = 0$

⑤ $A = Q^c$ A/2012 Qno5(b) $\quad m(Q^c) = \infty$

A/2015 Qno5(b) $\quad$ A/2012 Qno5(b) $\quad [1, 4)$

⑥ $A = (-3, 4) \cup [1, 6] = m((-3,4)) + m((1,6)) - m((-3,4) \cap [1,6])$

not disjoint $= 7 + 5 - 3$

$= 9$

$$\mathbb{R} = Q \cup Q^c$$

$m(\mathbb{R}) = m(\mathbb{R} \cap Q) + m(\mathbb{R} \cap Q^c)$

$\infty = m(Q) + m(Q^c)$

$= 0 + m(Q^c)$

$m(Q^c) = \infty.$

24th Nov, 2017. ✓ 123

Recall

· Lebesgue measurable sets (collection) $\mathcal{M}$ define on $2^{\mathbb{R}}$

· Lebesgue measure $m^*|_{\mathcal{M}} = m$.

Countably additive.

Proposition. A/2015 (Q no 5 (b))

If E_1 and E_2 are measurable sets

Then $m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$

Proof.

Given that E_1 is measurable

For any $A \subseteq \mathbb{R}$.

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$$

Set $A = E_2$

$$m^*(E_2) = m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1^c)$$

Since all sets are measurable we can replace

$m^* = m$

$$m(E_2) = m(E_2 \cap E_1) + m(E_2 \cap E_1^c) \quad (1)$$

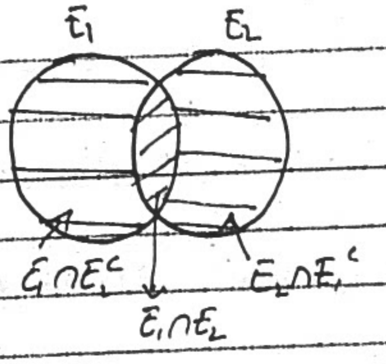
Replacing E_1 and E_2 in eq (1)

$$m(E_1) = m(E_1 \cap E_2) + m(E_1 \cap E_2^c) \quad (2)$$

$$m(E_1) + m(E_2) = 2m(E_1 \cap E_2) + m(E_2 \cap E_1^c) + m(E_1 \cap E_2^c)$$

(3)

Also; $E_1 \cup E_2$ can be expressed as disjoint union of the form



$E_1 \cup E_2 = (E_1 \cap E_2^c) \cup (E_2 \cap E_1^c) \cup (E_1 \cap E_2)$

Since m (Lebesgue measure) is countably additive

$m(E_1 \cup E_2) = m(E_1 \cap E_2^c) + m(E_2 \cap E_1^c) + m(E_1 \cap E_2)$

$m(E_1 \cap E_2^c) + m(E_2 \cap E_1^c) = m(E_1 \cup E_2) - m(E_1 \cap E_2)$ ④

put ④ in ③ we have

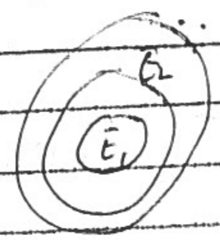
$m(E_1) + m(E_2) = 2m(E_1 \cap E_2) + m(E_1 \cup E_2) - m(E_1 \cap E_2)$

Thus,

$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$.

Def.

A countable collection of sets $\{E_k\}$ is said to be an ascending collection if for each k, $E_k \subseteq E_{k+1}$



A countable collection of sets $\{E_k\}$ is said to be an descending collection if for each k, $E_{k+1} \subseteq E_k$.

σ-algebra.

Theorem A/2015 Q.no.6(a)

The Borel σ-algebra $\mathcal{B}$ is generated by each of the following classes of sets.

1- The collection C_1 of all closed subsets of $\mathbb{R}$.

2- The collection C_2 of all intervals of the form $(-\infty, b]$.

3- The collection C_3 of all intervals of the form $(a, b]$.

[illegible] $\mathcal{B}$.

Proof.

~~Let~~ Let $\mathcal{B}$ be the Borel σ-algebra.

σ, Smallest [illegible]

Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be the σ-algebras generated by the classes of sets C_1, C_2 and C_3 respectively.

We need to prove that

$$\mathcal{B} \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_3 \supseteq \mathcal{B}$$

Hence $\mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$

By definition $\mathcal{B}$ contains open sets of $\mathbb{R}$ and is closed under complementation, therefore it contains closed subsets of $\mathbb{R}$.

$\Rightarrow \quad \mathcal{B}_1 \subseteq \mathcal{B}$

Next, we can write the set of the form

$(-\infty, b] \in \mathcal{B}_2$ and $(b, \infty)^c \in \mathcal{B}_1$

Thus, $(b,\infty)^c$, being complement of open set, is closed, and hence

$(b, \infty)^c \in \mathcal{B}_1$

$\Rightarrow \quad \mathcal{B}_2 \subseteq \mathcal{B}_1$

Next, we show that $\mathcal{B}_3 \subseteq \mathcal{B}_2$

$-\infty < a < b < \infty$

$\underset{\in \mathcal{B}_3}{(a, b]} = \underset{\in \mathcal{B}_2}{(-\infty, b]} \cap \left((-\infty, a]\right)^c \quad (a,\infty)$

$(a,b] = (-\infty, b] \cap (a,\infty)$

$(-\infty, a]^c$

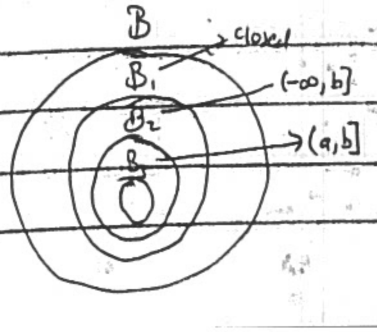
Since $\mathcal{B}_2$ is a σ-algebra Therefore $(a,b] \in \mathcal{B}_2$

$\Rightarrow \quad \mathcal{B}_3 \subseteq \mathcal{B}_2$

Finally, $\underset{\in \mathcal{B}}{(a,b)} = \bigcup_n \underset{\in \mathcal{B}_3}{\left(a, b - \frac{1}{n}\right]}$

Hence $\mathcal{B} \subseteq \mathcal{B}_3$

Result: Always generate unique structure called Borel σ-algebra



22nd Dec, 2017 ✓

Measurable functions.

Background.

In previous section, we have established the existence of non-mble sets. An important question arises, of what type are mble sets?

It will be seen that mble sets can also be characterized through mble functions.

Also, Lebesgue integration will be defined for a much larger class functions (mble functions) rather than continuous functions (used in Reimann).

Proposition. Qno7(a). A/2017, [redacted], [redacted], Qno6(b). A/2013, ~~A/2012 (old)~~

Let f be an extended real valued function defined on a mble (domain) set E. Then the following statements are equivalent.

$f^{-1}((\alpha,\infty]) = \{x \in E \mid f(x) > \alpha\}$.

(i) $\{x \in E \mid f(x) > \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$.

(ii) $\{x \in E \mid f(x) \geq \alpha\}$ " " "

iii) $\{x \in E \mid f(x) < \alpha\}$ " " "

iv) $\{x \in E \mid f(x) \leq \alpha\}$ " " "

Each of these statement implies that for each $\alpha \in \bar{\mathbb{R}}$, the set

$$\{x \in E \mid f(x) = \alpha\} \text{ is mble}$$

(i) $\Rightarrow$ (ii)

Given that $\{x \in E \mid f(x) > \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$

(~~Note that we can~~)

We prove that

$$\{x \in E \mid f(x) \geq \alpha\} = \bigcap_{n \geq 1}^{\infty} \{x \in E : f(x) > \alpha - \frac{1}{n}\}$$

let $y \in \{x \in E : f(x) \geq \alpha\}$

$$\Rightarrow f(y) \geq \alpha \quad \forall \alpha \in \mathbb{R}$$

$$f(y) \geq \alpha - \frac{1}{n} \quad \forall \alpha \in \mathbb{R}, \ \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \bigcap_{n \in \mathbb{N}} \{x \in E : f(x) > \alpha - \frac{1}{n}\} \quad \text{--- (I)}$$

Conversely, let

$$z \in \bigcap_{n \in \mathbb{N}} \{x \in E : f(x) > \alpha - \frac{1}{n}\}$$

$$\Rightarrow f(x) > \alpha - \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

$$f(x) \geq \alpha \quad \forall \alpha \in \mathbb{R}$$

$$\Rightarrow z \in \{x \in E : f(x) \geq \alpha\} \quad \text{--- (II)}$$

From (i) and (ii)

Hence, $\{x \in E : f(x) \geq \alpha\}$, being countable intersection of mble sets is mble

ii ⇒ iii,

Note that $\{x \in E : f(x) < \alpha\} = (\{x \in E | f(x) \geq \alpha\})^c$

Then $\{x \in E : f(x) < \alpha\}$ is ~~countable~~ measurable

(iii ⇒ iv)

$$\{x \in E : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < \alpha + \frac{1}{n}\}.$$

↓ give mble.

Measurable function

A/2013

Let f be an extended real valued function on a set E.
Then f is said to be mble function if E is mble and it satisfies one of the 5 statements

i) $\{x \in E : f(x) > \alpha\}$ is measurable for each $\alpha \in R$.

ii) $\{x \in E : f(x) \geq \alpha\}$ " " "

iii) $\{x \in E : f(x) < \alpha\}$ " " "

iv) $\{x \in E : f(x) \leq \alpha\}$ " " "

v) $\{x \in E : f(x) = \alpha\}$ " " " $\alpha \in \bar{R}$.

Algebra of Measurable function.

Theorem.

Let f and g be two mble functions defined on the same mble domain E and c be any real no. Then the functions

Q no 6(b) A/2011

i) $f+c$ ii) cf, iii) $f+g$, iv) $f-g$

A/2011 A/2011

v) f^2 vi) fg, vii) f/g, $g \neq 0$,

A/2011

viii) $f \vee g$, ix) $f \wedge g$ (x) $|f|$,

are mble.

Proof.

$f+c$ is mble

Consider

$$A = \{x \in E : (f+c)(x) > \alpha\}$$

$$= \{x \in E : f(x) + c > \alpha\}$$

$$= \{x \in E : f(x) > \alpha - c\}$$

Thus A is mble.

Consequently $f+c$ is mble

ii) cf is mble

Consider
$$A = \{x \in E : (cf)(x) > \alpha\}$$

Case I $c = 0$

(a) When $\alpha \geq 0$.
$A = \phi$ is mble.

(b) When $\alpha < 0$.
$A = E$ is mble.

Case II. $c > 0$.
$$A = \{x \in E : f(x) > \frac{\alpha}{c}\}$$

Since f is mble function, therefore by def, A is mble

Case III $c < 0$, or $-f$ is also mble
$$A = \{x \in E : f(x) < \frac{\alpha}{c}\}$$

Since f is mble function, therefore by def, A is mble

11th Jan, 2018.

iii) $f+g$.

To show that $f+g$ is measurable, we need to show that the set the
$$\{x \in E : (f+g)(x) > \alpha\} \text{ is measurable}$$

or

$\{x \in E : f(x) + g(x) > \alpha\}$ is measurable

Consider

$$f(x) + g(x) > \alpha$$

or (real, real, real)

$$f(x) > \alpha - g(x).$$

Using density theorem, there is a rational number, q, s.t

$$f(x) > q > \alpha - g(x)$$

The proof will follow if (for countable union are rationals)

$$\{x \in E : f(x) + g(x) > \alpha\} = \bigcup_{q \in Q} \left(\{x \in E : f(x) > q\} \cap \{x \in E : \alpha - g(x) < q\} \right)$$

(Mble under f, under g)

Let $y \in \{x \in : f(x) + g(x) > \alpha\}$

$$\Rightarrow \quad f(y) + g(y) > \alpha$$

$\exists$ q s.t $f(y) > q > \alpha - g(y)$

(i) (ii) satisfy of condition of just q

$\Rightarrow$

$$y \in \{x \in E : f(x) > q \text{ and } \alpha - g(x) < q\}.$$

(imp)

$$\Rightarrow y \in \{x \in E : f(x) > q\} \cap \{x \in E : \alpha - g(y) < q\}$$

$$\Rightarrow \quad y \in \bigcup_{q \in Q} \left(\{x \in E : f(x) > q\} \cap \{x \in E : \alpha - g(x) < q\} \right) \quad \text{--- (1)}$$

Let

$$z \in \bigcup_{q \in \mathbb{Q}} \left(\{x \in E : f(x) > q_0\} \cap \{x \in E : \alpha - g(x) < q_0\} \right)$$

$\Rightarrow$

$z \in \{x \in E : f(x) > q_0 \text{ and } \alpha - g(x) < q_0\}$, for some $q_0 \in \mathbb{Q}$.

$\Rightarrow z \in \{x \in E : f(x) > q_0\}$ and

$z \in \{x \in E : \alpha - g(x) < q_0\}$

$\Rightarrow \quad f(z) > q_0$ and $\alpha - g(z) < q_0$

$\Rightarrow \quad f(z) > q_0 > \alpha - g(z)$

or $\quad f(z) > \alpha - g(z)$

or $\quad f(z) + g(z) > \alpha$

$\Rightarrow \quad (f+g)(z) > \alpha$

$\Rightarrow \quad z \in \{x \in E : f(x) + g(x) > \alpha\}$ — ②

From ① and ②' it implies that the set $\{x \in E : (f+g)(x) > \alpha\}$, being countable union of mble sets is also mble

$-f$

Using result 2 of $c = -1$, then $-f$ is also mble.

iv) $g-f$ is mble.

Since g is mble (given) and $-f$ is also mble, therefore using (iii)

$g+(-f) = g-f$ is also mble.

v) f^2 is also mble.

Consider the set

$\{x \in E : f^2(x) > \alpha\}$ is mble.
all +ves

Case 1. When $\alpha < 0$.

Then

$\{x \in E : f^2(x) > \alpha\} = E$.

Since E is mble (given), therefore $\{x \in E : f^2(x) > \alpha\}$ is measurable

Case 2 : When $\alpha > 0$.

Then

$\{x \in E : f^2(x) > \alpha\} = \{x \in E : f(x) > \sqrt{\alpha}\} \cup \{x \in E : f(x) < -\sqrt{\alpha}\}$

Since f is mble, $\{x \in E : f^2(x) > \alpha\}$, being union of two mble sets is also mble

vi) $fg = f(x)g(x)$ is measurable

$$fg = \frac{1}{4}\{(f+g)^2 - (f-g)^2\}$$

Thus fg, being difference of two mble functions is also mble.

viii) $(f \vee g)(x) = \max(f(x), g(x))$ is also mble.
We have to show that

$= \{x \in E : (f \vee g)(x) > \alpha\}$ is mble.

$= \{x \in E : \max(f(x) \text{ or. } g(x)) > \alpha\}$ f(x) or g(x) is max.

$= \{x \in E : f(x) > \alpha\} \cup \{x \in g(x) > \alpha\}$

Hence $(f \vee g)(x)$ is mble.

ix) $(f \wedge g)(x) = \min(f(x), g(x))$.

$\{x \in E : (f \wedge g)(x) > \alpha\}$

$= \{x \in E : \min(f(x) \text{ and } g(x)) > \alpha\}$

$= \{x \in E : f(x) > \alpha\} \cap \{x \in E : g(x) > \alpha\}$

Hence $f \wedge g$ is mble

12th Jan, 2018.

x) $|f|$

constant function

$|f| = f^+ + f^-$ $\qquad$ $f^+ = \max(f(x), 0) = f \vee 0$

$f^+ = f \vee 0$ is measurable $\qquad$ $f^- = \max(-f(x), 0) = f \wedge 0$

$f^- = (-f) \vee 0$ is measurable

$\Rightarrow f^+ + f^-$ is mble.

$\Rightarrow |f|$ is mble.

viii) g/f is mble $\quad f(x) \neq 0, \ \forall x \in E.$

$g/f = g\left(\frac{1}{f}\right)$

We need to show that $1/f$ is mble

i.e., $\left\{x \in E : \frac{1}{f(x)} > \alpha\right\} \quad \forall \alpha \in \mathbb{R}.$

~~Case I~~

$$\left\{x \in E : \frac{1}{f(x)} > \alpha\right\} = \begin{cases} \{x \in E : f(x) > \alpha\} ; & \alpha = 0 \\ \{x \in E : f(x) > 0\} \cap \{x \in E : f(x) < \frac{1}{\alpha}\} ; & \alpha > 0 \\ \{x \in E : f(x) > 0\} \cup \left[\{x \in E : f(x) < 0\} \cap \{x \in E : f(x) < \frac{1}{\alpha}\}\right] & \alpha < 0 \end{cases}$$

for +ve: if $f(x)$ is +ve $\qquad$ if $f(x)$ is -ve

تاکہ [illegible] 0 سے بڑا رہے نیں

case and [$f(x)$ پے کر $\alpha > 0$] ہے -ve ہے کو leave کر دیں condition $\frac{1}{f(x)} > \alpha$ کیونکہ [illegible] satisfy نہیں ہو گی

In other words

~~$\{x \in E : f(x) < 0\} = \phi$~~

Thus, $\{x \in E : \frac{1}{f(x)} > \alpha\}$ is also measurable

Hence $g\left(\frac{1}{f}\right) = \frac{g}{f}$ is also measurable.

(mble) $f \Rightarrow |f|$ is mble

$|f|$ mble $\not\Rightarrow f$ is mble

Proposition

A real valued function f is measurable if and only if f^+ and f^- are measurable.

Proof.

Given f is mble

we have $f^+ = (f \vee 0)$ is mble

$f^- = (f \wedge 0)$ is mble

Conversely, let, f^+ and f^- are measurable

Then

$f = f^+ - f^-$ is also measurable.

Q no 1(b), A/2011 on E

Theorem Let f be an extended real-valued mble function. Let A be a mble subset of E. Then restriction of f to A is also mble $(f|_A)$

Proof.

We have to show that

$\{x \in A : f(x) > \alpha\}$

$= \{x \in E : x \in A \text{ and } f(x) > \alpha\}$

$= \{x \in E : f(x) > \alpha\} \cap A$ [illegible]

is mble $\forall \alpha \in \mathbb{R}$.

Thus $f|_A$ is also mble.

Since $\{x \in A : g(x) > \alpha\} \subseteq A$
By monotonicity.

$$m(\{x \in A : g(x) > \alpha\}) \leq m(A) = 0.$$

$$\Rightarrow m(\{x \in A : g(x) > \alpha$$

Consequently, $g(x)$ is mble on E.

Q no 7(b) → A/2018, A/2014 → Q no 6(b)
Theorem Let f and g be two mble functions defined on same mble domain E. Then the following sets are measurable.
~~Proof~~ 1) $\{x \in E : f(x) > g(x)\}$

2) $\{x \in E : f(x) < g(x)\}$

3) $\{x \in E : f(x) \leq g(x)\}$

4) $\{x \in E : f(x) \geq g(x)\}$.

Proof.

The proof will follow if we show that $\{x \in E : f(x) > g(x)\}$

$$= \bigcup_{q \in \mathbb{Q}} \left[\{x \in E : f(x) > q\} \cap \{x : g(x) < q\}\right]$$

Let $y \in \{x \in E : f(x) > g(x)\}$.

$\Rightarrow f(y) \notin g(y)$

$\Rightarrow \exists\, q_0 \in Q$ s.t $f(y) > q_0 > g(y)$

$\Rightarrow f(y) > q_0$ and $g(y) < q_0$

$\Rightarrow y \in \{x : f(x) > q_0\} \cap \{x : g(x) < q_0\}$

Thus

$y \in \bigcup_{q_0 \in Q} \left[\{x : f(x) > q_0\} \cap \{x : g(x) < q_0\}\right]$

$\{x \in E : f(x) > g(x)\} \subseteq \bigcup_{q_0 \in Q} \left[\{x : f(x) > q_0\} \cap \{x : g(x) < q_0\}\right]$

Conversely,

let

$z \in \bigcup_{q \in Q} \left[\{x : f(x) > q_0\} \cap \{x : g(x) < q\}\right]$ — (1)

Then there exist q_1 s.t

$z \in \{x : f(x) > q_1\} \cap \{x : g(x) < q_1\}$

$\Rightarrow z \in \{x : f(x) > q_1 \text{ and } g(x) < q_1\}$

$\Rightarrow f(z) > g(z)$

$\Rightarrow z \in \{x : f(x) > g(x)\}$ — (2)

$\bigcup_{q \in Q} \left[\{x : f(x) > q\} \cap \{x : g(x) < q_1\}\right] \subseteq \{x \in E : f(x) > g(x)\}$ — (2)

From (1) and (2).

$\{x \in E : f(x) > g(x)\} = \bigcup_{q \in Q} \left[\{x \in E : f(x) > q\} \cap \{x \in E : g(x) < q\}\right]$

Hence the set $\{x \in E : f(x) \geq g(x)\}$ is measurable.

اگر paper میں یہ پورا Theorem آئے تو ایک پارٹ کو اسطرح کرتے proper
Argument دوسرے کو ختم کرتا ہے اگر separately کوئی ایک پارٹ آجائے تو ایسے
thoroughly proof کرنا ہے

(2) Same procedure can be adopt to check the measurability of the set

$$\{x \in E : f(x) < g(x)\}$$

(3) Let f and g be two measurable functions defined on the same mble domain E.

Then ~~the following sets~~ we are to show that the set $\{x \in E : f(x) \leq g(x)\}$ is mble.

Let $A = \{x \in E : f(x) \leq g(x)\}$.

we find

$$A^c = \{x \in E : f(x) > g(x)\},$$ ~~which~~ is mble set (part 2).

Consequently, $A = \{x \in E : f(x) \leq g(x)\}$ is mble.

(4) Let f and g be two measurable functions defined on the same mble domain E.

We are to show that the set

$$A = \{x \in E : f(x) \geq g(x)\}$$ is mble.

Note that

$$A^c = \{x \in E : f(x) < g(x)\}$$ is mble (part 2).

Hence, A is mble.

19th Jan, 2018.

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Step Function.

A/2014 Q no 7(b)

Let ϕ be a real valued function defined on $[a,b]$. We say that ϕ is a step function if there is a partition on $[a,b]$, $a = x_0 < x_1 < x_2 \dots < x_n = b$ s.t. ϕ assumes only one value a_i in the interval (x_{i-1}, x_i), $i = 1, 2, 3, \dots, n$

Step function is a constant function in each interval (x_{i-1}, x_i).

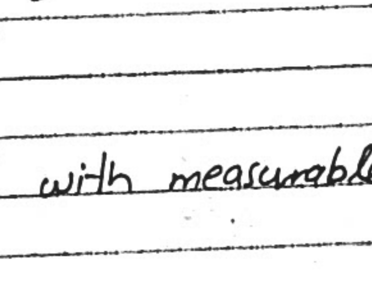
Congruents '[' or '('

A/2014 Q no 7(b)

Theorem A step function f with measurable domain is measurable.

Proof.
Let f assumes finite number of values $a_1, a_2, \dots, a_n$ on domain $[a,b]$.
We need to show that the set
$\{x \in [a,b] : f(x) > \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

α is min. or max.



Then there are three possible cases for α
Case I If $\alpha < \min(a_1, a_2, \dots, a_n)$
Then
$\{x \in [a,b] : f(x) > \alpha\} = [a,b]$.

$\alpha < \min(a_1, \dots a_n)$

Write Arg

Case II. If $\alpha \geq \max(a_1, a_2, a_3, \dots, a_n)$.
Then
$$\{x \in [a,b] : f(x) > \alpha\} = \phi.$$

Write arg.

Case III If $\min(a_1, a_2, \dots a_n) \leq \alpha < \max(a_1, a_2, \dots, a_n)$.

Then $\{x \in [a,b] : f(x) > \alpha\}$ is ~~the~~ atleast union of the interval $[x_i, x_{i+1})$ s.t. $f(x) = a_j$ in $[x_i, x_{i+1})$.
and
$$\alpha < a_j \leq \max(a_1, a_2, a_3, \dots, a_n)$$
$$\{x \in [a,b] : f(x) > \alpha\} = \bigcup_j [x_j, x_{j+1})$$ in
such that
$$f(x) = a_j > \alpha \quad \forall x \in [x_j, x_{j+1})$$

Write Arg.

9th Feb, 2018.

Past Paper

Example

Evaluate Lebesgue integral of the following function. A|2017 Q no 8 (a)(ii).

$f : [0,1] \to \{0,1\}$.

$$f(x) = \begin{cases} 1 & ; x \in Q \cap [0,1] \\ 0 & ; x \in Q' \cap [0,1]. \end{cases}$$

$f(x) = 1 \cdot \chi_A + 0 \cdot \chi_B$

$\int_E f(x) = 1 \cdot m(A) + 0 \cdot m(B)$, $m(B) \le m(E) < \infty$ finite

Note that $m(A) = 0$, A being countable subset of E

$= 0$ is Lebesgue integrable

But not Rieman Integrable (Prove it)

A|2017 $f : (-4,5) \to \mathbb{R}$ Q no 8 (a)(i)

$$f(x) = \begin{cases} -4 & ; x \text{ is rational in } (-4,5) \\ 8 & ; x \text{ is irrational in } (-4,5). \end{cases}$$

Let $A = Q \cap (-4,5)$, $B = Q' \cap (-4,5)$

$f(x) = -4 \cdot \chi_A + 8\chi_B$

$\int_{(-4,5)} f(x) = -4\, m(A) + 8\, m(B)$

$= -4(0) + 8(9)$ $\qquad m(A) = 0$

$= 72$

Because 148

$(-4,5) = A \cup B$

$B = (-4,5) | A$

$m(B) = m\{(-4,5) | A\}$

Using excision property

$m(B) = m((-4,5)) - m(A)$

$= 5+4 = 0$

$= 9.$

4

$f: [0,1] \longrightarrow \{0,1\}$

$$f(x) = \begin{cases} 1 & x \neq 1/2 \\ 0 & x = 1/2. \end{cases}$$

A/2016 Q.no 8(b)(ii)

5. If A and B be disjoint subsets of E, then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof.

Consider

$$\int_{A\cup B} f = \int_{A\cup B} f \cdot \chi_{A\cup B}$$

$$= \int_{A\cup B} f(\chi_A + \chi_B)$$

$$= \int_{A\cup B} f\chi_A + \int_{A\cup B} f\chi_B$$

$$= \int_A f\chi_A + \int_B f\chi_B$$

$$= \int_A f + \int_B f.$$

A/2015 Q.no 8 b(i)

6. $\left|\int_E f\right| \le \int_E |f|$

Paste here $\int af + bg = a\int f + b\int g$.

Proof. Case I $f \geq 0$ on E

$f \geq 0 \Rightarrow \int_E f \geq 0$

1502

$\Rightarrow \left|\int_E f\right| = \int_E f \rightarrow (1)$

Also

$f \geq 0 \Rightarrow$ ~~$|f| \geq 0$~~ $\Rightarrow |f| = f$

$\Rightarrow \int_E |f| \geq 0 = \int_E f$ — (2)

~~$\int_E f \geq 0$~~ f (2)

Comparing (1) and (2)

$\left|\int_E f\right| = \int_E |f|$

Case II if $f(x) < 0$ $\forall$ all values on E

$\int_E f < 0$

$\left|\int_E f\right| = -\int_E f \rightarrow (3)$

Also

$f(x) < 0$

$|f(x)| = -f(x)$

$\int_E |f| = -\int_E f \rightarrow (4)$

Comparing (9) and (5)

$\left|\int_E f\right| = \int_E |f|$

Case III if $f(x)$ assumes both negative and non-negative values on E, then we can define

$$A = \{x \in E : f(x) \geq 0\}$$

$$B = \{x \in E : f(x) < 0\}$$

$$E = A \cup B \qquad A \cap B = \phi$$

(We can write)

$$\int_E f = \int_{A \cup B} f = \int_A f + \int_B f$$

$$\left|\int_E f\right| = \left|\int_A f + \int_B f\right|$$

$$\left|\int_E f\right| \leq \left|\int_A f\right| + \left|\int_B f\right|$$

$$= \int_A |f| + \int_B |f|$$

Case I   Case II

$$= \int_{A \cup B} |f|$$

$$= \int_E |f|$$

$$\Rightarrow \left|\int_E f\right| \leq \int_E |f|$$

Assignment 152

Pg 195 (Local Book) Proposition

12 Feb, 2018.

Proposition. A/2014 Q.no.8(a).

Let f be a bounded real valued function defined on a mble set E, $m(E) < \infty$. Then

f is Lebesgue integrable $\Leftrightarrow$ f is mble.

$$\inf_{\psi \geq f} \int_E \psi = \sup_{\varphi \leq f} \int_E \varphi \quad \text{iff } f \text{ is mble.}$$

where ψ and φ are simple functions.

Proof.

(i) Let f is mble. We need to show that f is Lebesgue integrable.

Simple Approx. lemma. (derivation)

Then, for any given $n \in \mathbb{N}$, by Simple Approx. Lemma there are simple functions φ_n and ψ_n such that

$$\varphi_n \leq f \leq \psi_n \quad \text{and}$$

$$0 \leq \psi_n - \varphi_n < \frac{1}{n} \equiv \epsilon$$

By Montonicity of Lebesgue integral for bounded function implies that

$$\int_E (\psi_n - \varphi_n) < \int_E \frac{1}{n} \cdot 1$$

By Linearity property

$$0 \le \int_E \psi_n - \int_E \varphi_n < \frac{1}{n} m(E) \longrightarrow (1)$$

By definition of infimum and supremum.

$$\inf_{\psi \geq f} \int_E \psi \le \int_E \psi_n \quad \text{--- (2)}$$

$$-\sup_{\varphi \le f} \int_E \varphi \le -\int \varphi_n \longrightarrow (3)$$

(2) + (3)

$$0 \le \inf_{\psi \geq f} \int_E \psi - \sup_{\varphi \le f} \int_E \varphi \le \int_E \psi_n - \int_E \varphi_n$$

(1) $\Rightarrow$

$$0 \le \inf_{\psi \geq f} \int_E \psi - \sup_{\varphi \le f} \int_E \varphi < \frac{1}{n} m(E) \quad \text{--- (4)}$$

Since (4) hold for all $n \in \mathbb{N}$, it is only possible if

$$\inf_{\psi \geq f} \int_E \psi - \sup_{\varphi \le f} \int_E \varphi = 0$$

$$\inf_{\psi \geq f} \int_E \psi = \sup_{\varphi \le f} \int_E \varphi .$$

Conversely, Suppose f is integrable i.e.,

$a = \inf_{\psi \geq f} \int_E \psi = \sup_{\varphi \leq f} \int_E \varphi$ for ~~any~~ simple function ψ and φ.

We need to show that f is measurable.
By definition of infimum and supremum, for any given $\epsilon = \frac{1}{2n} > 0$, there are simple functions φ_n and ψ_n s.t.,

$$a + \frac{1}{2n} > \int_E \psi_n$$

$$a - \frac{1}{2n} < \int_E \varphi_n.$$

Then,

$$\int_E \psi_n - \int_E \varphi_n < a + \frac{1}{2n} - a + \frac{1}{2n}$$

$$\int (\psi_n - \varphi_n) < \frac{1}{n} \longrightarrow (1)$$

Let us define

$$\psi^* = \inf_n \psi_n$$

$$\varphi^* = \sup_n \varphi_n.$$

Then

$$\varphi^* \leq f \leq \psi^*$$

Since ψ_n and φ_n are all mble, therefore ψ^* and φ^* are also mble.

The proof will follow if we show that

$$\varphi^* = \psi^* \quad a.e.$$

For this purpose let us define

$$A = \{x \in E : \psi^*(x) \neq \varphi^*(x)\}$$

(or ~~we can~~) $\psi^* \geq \varphi^*$ therefore we can write

$$A = \{x \in E : \psi^*(x) > \varphi^*(x)\}$$

Note that A is mble. We show that

$$m(A) = 0.$$

Similarly we can define

$$A_k = \{x \in E : \psi^*(x) > \varphi^*(x) + \tfrac{1}{k}\}, \ k \in N$$

Then A_k is also mble.

$$A = \bigcup_{k=1}^{\infty} A_k.$$

Let $x \in A$, then $\psi^*(x) > \varphi^*(x)$ or

$$0 < \psi^*(x) - \varphi^*(x)$$

let

$$\psi^*(x) - \varphi^*(x) = \epsilon$$

$$\Rightarrow \quad 0 < \psi^*(x) - \varphi^*(x) = \epsilon$$

Then we can find $n^* \in N$ s.t

$$1 \leq n^* \epsilon$$

or $\frac{1}{n^*} \leq \epsilon$

$\therefore \frac{1}{n^*} < \psi^*(x) - \varphi^*(x)$

or

$\frac{1}{n^*} + \varphi^*(x) < \psi^*(x)$

$\Rightarrow x \in A_{n^*}$

$\Rightarrow x \in \bigcup_{k=1}^{\infty} A_k$ for some k. $\longrightarrow$ (2)

Let $x \in \bigcup_k A_k$

then $x \in A_{n_0}$ for $n_0 \in \mathbb{N}$

$\Rightarrow \psi^*(x) > \varphi^*(x) + \frac{1}{n_0}$

f msble $\to A \to A_k \to B_{k_n}$

$\Rightarrow \psi^*(x) > \varphi^*(x)$

$\Rightarrow x \in A$ —— (3)

(2) and (3) implies

$A = \bigcup_k A_k$

Now we only need to show that $m(A_k) = 0$

For this let us define

$B_{k_n} = \{x \in E : \psi_n^*(x) > \varphi_n(x) + \frac{1}{k}\} \subseteq E$

It can be observed that

$A_k \subseteq B_{k_n} \quad \forall k$ (Prove it)

By def of B_{n_k}

$\varphi_n(x) - \varphi_n(x) > \frac{1}{k}$

Monotonocity

$$\int_E (\psi_n - \varphi_n) > \int \frac{1}{k} \chi_{B_{kn}}$$

$$\int_E \psi_n - \int_E \varphi_n > \frac{1}{k} m(B_{n_k})$$

① ⇒

$$\frac{1}{n} > \frac{1}{k} m(B_{kn})$$

or $\quad m(B_{kn}) < \frac{k}{n}$ ——— ④

inequality hold for all n

⇒ $\quad m(B_{kn}) = 0$ as $n \to \infty$

Since $A_k \subseteq B_{kn}$

$$m(A_k) < \frac{k}{n}$$

$= 0$ as $n \to \infty$

Assigment Pg 158. ✓ 158

Since $A = \bigcup_k A_k$

$$m(A) = m\left[\bigcup_k A_k\right]$$

$$= \sum_k m_i(A_k)$$

$$m(A) = 0$$

$\Rightarrow \varphi^* = \varphi_1^*$ a.e

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Q no 9(a) A/2016, Q no 8(b) A/2012, Q no 8(b) A/2011, Q 8(a) 2013

Bounded Convergence Theorem.

Let $\{f_n\}$ be a sequence of uniformly pointwise bounded, mble function on E $(m(E)<\infty)$.

That is, $\exists$ M s.t

سب کے لیے bound ایک ہی ہو — not individually bounded

$$|f_n| \leq M \quad \forall n \in N.$$

If $\lim_{n\to\infty} f_n(x) = f(x)$ for each x then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof [illegible] U.C Case I [illegible] Paper

Case I (Uniform Convergence).

Case II: Pointwise convergence

Suppose that $\{f_n\}$ converges point wise to f.

Given that $|f_n| \leq M \quad \forall n \in N$

Then Clearly

$$\left|\lim_{n\to\infty} f_n\right| \leq M$$

$$\Rightarrow |f| \leq M$$

Using (previous result), for $\epsilon>0$ and $\sigma>0$ $\exists$ a mble set A and $p \in N$ s.t

$$m(A) < \sigma = \frac{\epsilon}{4M} \qquad \text{set } \sigma = \frac{\epsilon}{4M}$$

$f_1(x)\ f_2(x)$

Set $\epsilon = \epsilon / 2m(E)$. 160

and $|f_n(x) - f(x)| < \frac{\epsilon}{2m(E)}$ $\forall\ n \geq p,\ x \in A^c$ (or $x \notin A$) — (2)

Consider

$$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right|$$

$$\leq \int_E |f_n - f|.$$

$$= \int_A |f_n - f| + \int_{A^c} |f_n - f|.$$

(2) $\Rightarrow$

$$\left|\int_E f_n - \int_E f\right| \leq \int_A |f_n| + \int_A |f| + \int_{A^c} \frac{\epsilon}{2m(E)}$$

$$\leq \int_A M\cdot 1 + \int_A M\cdot 1 + \frac{\epsilon}{2m(E)}\, m(A^c)$$

$$= 2M\, m(A) + \frac{\epsilon}{2m(E)}\, m(A^c)$$

$$\leq \frac{2M\epsilon}{4M} + \frac{\epsilon}{2m(E)}\, m(E)$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon.$$

$$\Rightarrow \left|\int_E f_n - \int_E f\right| < \epsilon$$

$$\Rightarrow \lim_{n\to\infty} \int_E f_n = \int_E f$$

Examp...

theore...

of fun...

$f_n(x)$

$f_n(x) =$

$f_n(x) =$

$f_n(x) =$

n=1

$f_1(x) = 1$

n=2 = 1

$f_2(x)$

n=3

Now

$\lim_{n\to\infty} f_n$

In short

$$\int_E f = \sup_{h \le f} \int_E h$$

where h is a bdd, mble function of finite support.

Proposition A/2013 Q8(a)

Let f and g be non-negative mble function on E. Then

1) A/2010 $\int_E af = a\int_E f$ A/2011 Q no 8(a), Q no 8(a)

2) A/2013 $\int_E (f+g) = \int_E f + \int_E g$ Q no 8(a)

te support

3) A/2011 $f \le g \Rightarrow \int_E f \le \int_E g$. Q no 8(a)

4) Let h be a bdd mble function of finite support such that $h \le f$ - $ah \le af$

If $a = 0$, (trivial)

Then for $a > 0$,

$$\int_E af = \sup_{ah \le af} \int_E ah = a \sup_{h \le f} \int_E h = a\int_E f$$

2) $\int_E (f+g) = \int_E f + \int_E g$.

Let h and k be any arbitrary bdd mble function on E of finite support on E, such that $0 \le h \le f$ and $0 \le k \le g$.

Then

$h+k \le f+g$ and $h+k$ is a bdd mble function of finite support.

$$\int_E f+g = \sup_{h+k \le f+g} \int_E h+k$$

$$\geq \int_E h+k$$

$$\int_E f+g \geq \int_E h + \int_E k \quad \text{linearity property}$$

Since the above inequality holds for all h and k functions, therefore

$$\int_E (f+g) \geq \sup_{\substack{h \le f \\ k \le g}} \left[\int_E h + \int_E k \right]$$

$$= \sup_{h \le f} \int_E h + \sup_{k \le g} \int_E k$$

$$\int_E (f+g) \geq \int_E f + \int_E g \quad \text{--- (1)}$$

Conver…
functi…

Defi…

By c…
(a) If

(b) I…
The…
Then
k…
=>

Thus
=>

$\int_E l$…

$\int_E (f+g)$



Conversely, let l be a bounded mble function of finite support on E such that

$$0 \le l \le f+g.$$

Define the function h and k, as

$$h = \min(f, l) \qquad \min(f, f+g)$$

$$k = l - h$$

By definition, $h \le f$ (h, k are arbitrary)

(a) If $l \le f$

$\Rightarrow l = h$

$$k = l - l = 0 \le g$$

(b) If $l > f$

Then $h = f$

Thus

$$k = l - f \le f + g - f = g \quad \text{Thus } 0 < k \le g.$$

$\Rightarrow 0 < k \le g.$

Thus from $k = l - h \Rightarrow l = k + h.$

(Linear property for bdd mble functions)

$$\Rightarrow \int_E l = \int_E (k+h) = \int_E k + \int_E h.$$

$$\int_E l \le \sup_{k \le g} \int_E k + \sup_{h \le f} \int_E h$$

$$\int_E (f+g) \le \int_E g + \int_E f.$$

Since l was taken to be arbitrary and $\int_E g + \int_E f$ is upper bound

Thus

$$\sup_{l \leq f+g} \int_E l \leq \int_E f + \int_E g.$$

$$\int_E (f+g) \leq \int_E f + \int_E g \quad \text{--- (2)}$$

From (1) and (2).

$$\int_E (f+g) = \int_E f + \int_E g.$$

3. $f \leq g \Rightarrow \int_E f \leq \int_E g.$

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$\Rightarrow \quad \infty \leq \int_E f.$

$\Rightarrow \quad \int_E f = \infty.$

23rd Feb, 2018.

Application of Chebychev

Proposition Q/2017 Ph08(b)

Let f be a non-negative mble function on E. Then

$$\int_E f = 0 \quad \text{iff} \quad f = 0 \text{ a.e on } E.$$

Proof.

Let $\int_E f = 0$ then for any $n \in \mathbb{N}$

Using Chebychev inequality,

$$m\left(\left\{x \in E : f(x) \geq \frac{1}{n}\right\}\right) \leq n \int_E f$$

$$\Rightarrow m\left(\left\{x \in E : f(x) \geq \frac{1}{n}\right\}\right) \leq 0. \quad \text{for each } n.$$

Since m is non-negative

$$\Rightarrow m\left(\left\{x \in E : f(x) \geq \frac{1}{n}\right\}\right) = 0 \longrightarrow (1) \text{ for each } n$$

Note that $\{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} \left\{x \in E : f(x) \geq \frac{1}{n}\right\}$

By Countable sub additivity of m

$$m\left(\{x \in E : f(x) > 0\}\right) \leq \sum_n m\left(\left\{x \in E : f(x) \geq \frac{1}{n}\right\}\right)$$

(1) $\Rightarrow$

$m(\{x \in E : f(x) > 0\}) \leq 0. \Rightarrow m(\{x \in E : f(x) > 0\}) = 0.$

Let $A = \{x \in E : f(x) > 0\}$

Then

$$m(A) = 0.$$

باقی 0 والی نہیں تھی کیونکہ $f \geq 0$ ہے

It implies that $f = 0$ a.e on E.

Conversely,

Let $f = 0$ a.e on E

Let φ be a simple function and h be a bounded function of finite support such that

$$0 \leq \varphi \leq h \leq f = 0 \text{ a.e.}$$

Then $\varphi = 0$ a.e on E

$\Rightarrow \int_E \varphi = 0$ on E.

The above equality holds for all φ functions such that $0 \leq \varphi \leq h$

$\Rightarrow$

$$\sup_{\varphi \leq h} \int_E \varphi = 0.$$

$\Rightarrow \int_E h = 0.$

The above equality holds for all h functions such that $0 \leq h \leq f$

$$\sup_{h \leq f} \int_E h = 0.$$

$$\int f = 0$$

Fatou's Lemma D/2012 Q.no9(a)

Like Bounded convergence theorem.

Let $\{f_n\}$ be a sequence of non-negative mble functions on E. if

If:- $\{f_n\} \to f$ pointwise on E, then

$$\int \lim_{n\to\infty} f_n = \int_E f \leq \underline{\lim} \int_E f_n$$

$$\int_E \underline{\lim}\, f_n \leq \underline{\lim} \int_E f_n$$

$$\int_E \overline{\lim}\, f_n \leq \overline{\lim} \int_E f_n$$

Proof.

Let h be a bounded measurable function of finite support such that

$$0 \leq h \leq f \to ①$$

Since h is bounded, choose $M > 0$ s.t.

$|h| \leq M$ and

define a set

$E_0 = \{x \in E : h(x) \neq 0\}$ finite support.

Then $m(E_0) < \infty$.

Define a sequence $\{h_n\}$ of bounded functions

$$h_n = \min(h, f_n) \quad \text{non-negative} \qquad h \le f$$

Then h_n is a mble function and

$$|h_n| \le |M| \qquad \forall x \in E_0$$

whereas $|h_n| \le |h| \le M$

$$h_n = 0 \qquad \forall x \in E \setminus E_0$$

$$\lim_{n\to\infty} h_n = \min\left(\lim_{n\to\infty} h, \lim_{n\to\infty} f_n\right)$$

$$= \min(h, f)$$

$$= h \qquad \because h \le f$$

Apply bounded convergence theorem to $\{h_n\}$

$$\lim_{n\to\infty} \int_E h_n = \int_E h = \int_E \lim_{n\to\infty} h_n$$

Further,

Note that

$$h_n \le f_n$$

$$\int_E h_n \le \int f_n$$

$$\underline{\lim} \int_E h_n \le \underline{\lim} \int_E f_n$$

$$\lim_n \int_E h_n \le \underline{\lim} \int f_n$$

reorem.
-negative
function
condit.

$\int_E h \le \underline{\lim} \int f_n$ since it holds for all h

$\sup_{h \le f} \int_E h \le \underline{\lim} \int_E f_n$

$\int_E f \le \underline{\lim} \int_E f_n$

Example.

Let $E = (0,1]$

$$f_n = n \chi_{(0,\frac{1}{n})}$$

$f_n \to f = ?$

Archemedean property
$a > 0$, $b > 0$
$na > b$
$nx > 1$
$x > \frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+1}$

$\lim_{n\to\infty} f_n = \lim_{n\to\infty} n \chi_{(0,1/n)}$

$= 0$

Then

$f_n \to 0 = f$.

$\Rightarrow \int_E f = \int_E 0 = 0$.

Now

$\underline{\lim} \int f_n$

$\int_E f_n = \int_0 n \cdot \chi_{(0,1/n)}$

$= n \cdot m\left(\left[0, \frac{1}{n}\right]\right)$

$= \frac{n}{n}$

$= 1$

$\underline{\lim} \int_E f_n = 1$

Then

$\int_E f$

Alrol

Mon

Let

of n

$\lim_{n\to}$

$\int \lim_{n\to\infty} f_n =$

Since

conver

Theref

$\Rightarrow$

By

for all n

Then
$0 < 1$
$$\int_E f < \underline{\lim} \int f_n$$

Al2014 Qno8(b) Special case of Fatou's lemma.

Monotone Convergence Theorem.

Let $\{f_n\}$ be an increasing sequence of non-negative mble functions and

properly

>0

$\lim_{n\to\infty} f_n = f$ Then

$$\int \lim_{n\to\infty} f_n = \int_E f = \lim_n \int_E f_n$$

$\geq \frac{1}{n+1} \geq \frac{1}{n+1}$

if $\{x_n\} \uparrow$ and bdd above then $x_n \to \sup$.

if $\{x_n\} \downarrow$ and bdd below then $x_n \to \inf$.

Since any increasing and bounded sequence converges to its supremum and here

$$\lim_{n\to\infty} f_n = f$$

Therefore, f is the supremum of $\{f_n\}$
$\Rightarrow$ $f_n \leq f$
By monotonicity
$$\int_E f_n \leq \int_E f$$

$$\overline{\lim} \int_E f_n \leq \int_E f \longrightarrow (1)$$

But, Fatou's lemma implies

$$\int_E f \leq \underline{\lim} \int f \longrightarrow (2)$$

(1) and (2)

$$\overline{\lim} \int f_n \leq \int f \leq \underline{\lim} \int f_n.$$

But $\underline{\lim} \leq \overline{\lim}$

Thus $$\int_E f = \underline{\lim} \int_E f_n.$$

A/2014 V. imp

Example Q no. 6 (b)

Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{1}{n} \chi_{[n,\infty)}(x).$$

a) Find $\lim_{n\to\infty} f_n$

(b) Determine whethere M.C.T hold for $\{f_n\}$.

Sol

(a)

$a > 0$, $b > 0$

$na > b$

$$\lim_{n\to\infty} f_n = 0$$

Set $a = \frac{1}{x}$, $b = 1$

$$\frac{n}{x} > 1$$

$$n > x$$

$$x < n < n+1 < n+2$$

$$\int f = \int_E 0 = 0. \quad \text{—(1)}$$

$$\lim_n \int f_n = \lim_{n\to\infty} \int \frac{1}{n} \chi_{(n,\infty)}.$$

$$= \lim_{n\to\infty} \left(\frac{\infty}{n}\right) = \lim_{n\to\infty} (\infty).$$

$$= \infty. \quad \text{—(2)}$$

$$0 < \infty.$$

Corrolary

Let $\{u_n\}$ be the sequence of non-negative measurable functions and let

$$f = \sum_{n=1}^{\infty} u_n \text{ than}$$

relate it with sum of series = limit of seq

$$\int \sum_n u_n = \int_E f = \sum_n \int_E u_n$$

let

$$f_k = \sum_{k=1}^{k} u_n \quad \text{(partial sum)}$$

Then f_k is increasing seq. of non-negative measurable function, converging to f.

Thus, by using M.C. Theorem on $\{f_n\}$

Linearity of Integration.

Theorem. 1/12/2014 (Monday)

Let f and g be integrable over E, then for any $a \in \mathbb{R}$

$$\int_E af = a\int_E f$$

$$\int_E (f+g) = \int_E f + \int_E g$$

} condition of Subspace

i) $av \in V$

ii) $u+v \in V$

If we treat f and g as vector.

اصل میں ہم نے یہ ثابت کرنا ہے

Proof.

When $a=0$. trivial

over

Case II $a > 0$.

Since $f = f^+ - f^-$

$(af) = (af)^+ - (af)^-$

By def.

$(af)^+ = \max\{af(x), 0\} = a\max\{f(x), 0\} = af^+$

$(af)^- = \max\{-af(x), 0\} = a\max\{-f(x), 0\} = af^-$

~~By def of $(af)^+$~~

$\int_E (af) = \int_E (af)^+ - \int (af)^-$

$= \int af^+ - \int af^-$

$= a\int_E f^+ - a\int_E f^-$ f^+ and f^- are non-neg

$= a\left(\int_E f^+ - \int_E f^-\right)$

$= a\int f$

Case III $a < 0$.

Since $f = f^+ - f^-$

$(af) = (af)^+ - (af)^-$

By def

$(af)^+ = \max\{af(x), 0\} = a \cdot \min\{f(x), 0\}$

$= -a \max\{-f(x), 0\} = -af^-$

$(af)^- = \max\{-af(x), 0\} = -(-a)\max\{f(x), 0\}$

$= -af^+.$

Thus

$$\int_E (af) = \int_E (af)^+ + \int_E (af)^-$$

$$= \int_E -af^- - \int_E -af^+$$

$$= -a\int_E f^- + a\int_E f^+$$

$$= a\left(\int_E f^+ - \int_E f^-\right)$$

$$= a\int_E f.$$

1st March 2018. 177

Lebesgue Dominated Convergence

Theorem. Q.no 9(a) A|2015, Q.no 9(a) A|2013

Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that

$$|f_n| \leq g \quad \text{on } E \quad \forall n$$

If $\{f_n\} \rightarrow f$ pointwise a.e on E, then f is integrable over E, and

$$\lim_{n\rightarrow\infty} \int_E f_n = \int_E \lim_{n\rightarrow\infty} f_n = \int_E f$$

Generalized Dominated Convergence theorem (Do it!) local book

Proof.

Since $|f_n| \le g \quad \forall n$, and

$f_n \longrightarrow f$, therefore

$|f| \le g$.

By Monotonicity of Integration

$$\int_E |f| \le \int_E g.$$

But $\int_E g$ is integrable

$$\Rightarrow \int_E |f| < \infty.$$

$\Rightarrow \quad f$ is integrable. $\quad |f| \Leftarrow f$

Next,

$|f_n| \le g$

$\Rightarrow -g \le f_n \le g.$

$-g \le f_n \Rightarrow 0 \le g + f_n \quad , \quad f_n \le g \Rightarrow 0 \le g - f_n.$

Note that $\{g + f_n\}$ is a seq. of non-negative mble functions

$\lim_{n\to\infty} (g + f_n) = g + f \quad , \quad \lim_{n\to\infty} (g - f_n) = g - f$

$\Rightarrow \{g + f_n\} \longrightarrow g + f \quad , \quad \Rightarrow \{g - f_n\} \to g - f$

Applying Fatou's Lemma, we have

$$\int_E \lim_{n\to\infty} (g+f_n) \le \underline{\lim} \int_E (g+f_n).$$

$$\int_E g+f \le \underline{\lim}\left(\int_E g + \int_E f_n\right)$$

$$\int_E g + \int_E f \le \int_E g + \underline{\lim} \int_E f_n$$

Since g is integrable $\Rightarrow \int_E g < \infty$.

$$\int_E f \le \underline{\lim} \int_E f_n \longrightarrow (1).$$

Applying Fatou's lemma, we have

$$\int_E \lim_{n\to\infty}(g-f_n) \le \underline{\lim} \int_E (g-f_n).$$

$$\int_E g - \int_E f \le \int_E g - \overline{\lim} \int_E f_n.$$

Since $\int_E g < \infty$

$$-\int_E f \le -\overline{\lim} \int_E f_n$$

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$$\int_E f_n \geq \overline{\lim} \int_E f_n \longrightarrow (2)$$

Since
$\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$, therefore

(1) and (2) $\Rightarrow$

$$\int_E f \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E f.$$

$$\Rightarrow \underline{\lim} \int_E f_n = \overline{\lim} \int_E f_n = \int_E f$$

$$\lim_n \int_E f_n = \int_E f.$$

Example.
$E = (0, 1]$
$f_n(x) = n \chi_{[0, 1/n]}$.

$\lim_{n \to \infty} f_n(x) = 0,$

We cannot find dominated function g for $\forall\, n \in \mathbb{N}$. Do it!

Past Paper Questions.

A/2017.

Q.no.9 (a)

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions defined by

$$f_n(x) = \frac{1}{n} \chi_{[n,\infty)} \quad \forall\, n = 1, 2, 3, \ldots$$

Then prove that

$$\lim_n \int_E f_n \neq \int_E \lim_n f_n.$$

Sol.

Clearly

$f_n(x) = \frac{1}{n} \chi_{[n,\infty)}$ is a decreasing sequence of non-negative measurable functions with $\lim_n f_n = 0$.

Hence $\int \lim_n f_n = 0 \longrightarrow (1)$

$$\int f_n = \int \frac{1}{n} \chi_{[n,\infty)} = \frac{1}{n} \int \chi_{[n,\infty)}$$

$$= \frac{1}{n} m([n,\infty)) = +\infty.$$

So, $\lim_n \int f_n = +\infty$.

From ① and ②.

$$\int \lim_n f_n \neq \lim_n \int f_n$$

Qno 9 (b) does not found
Qno 5(b) " "

A/2016. 183 | A/2011 Q no 5(b), Q no 7(a) |

Q no 6 (b) | A/2014 Q no 6(a) | A/2013 Q no 6(a) |

Demonstrate the existence of a non-Lesbegue mble set P contained in $A = [0,1)$.

Proof. Let $A = [0,1)$. ~~and~~
We show that there exist a set $P \subset [0,1]$ of positive Lebesgue outer measure such that the translated set $E + q$, $q \in [-1,1] \cap Q$, form a pairwise disjoint covering of the interval $[0,1]$.
As the Lesbegue measure is translation invariant and countably additive on measurable sets, the set P cannot be Lebesgue measurable.

Define an equivalence relation on the real line by
$$x \sim y \Leftrightarrow x - y \in Q$$
The the relation '$\sim$' is an equivalence relation; as.

i) $x \sim x \Rightarrow x - x \in Q$
$\Rightarrow 0 \in Q$

ii) if $x \sim y$ i.e, $x - y \in Q$
then $y \sim x$ i.e, $y - x \in Q$

iii) Let $x \sim y$ and $y \sim z$ then there are ~~not~~ rational numbers $q_1, q_2 \in Q$ such that
$$x - y = q_1 \quad , \quad y - z = q_2$$
Now
$$x - z = x - y + y - z = q_1 + q_2 = q \in Q.$$
Thus
$$x \sim z$$

The equivalence relation '$\sim$' decomposes $\mathbb{R}$ into disjoint classes.

Denote the equivalence class containing 'x' by P_x.

Note that each equivalence class is countable (because Q is countable) and therefore since $\mathbb{R}$ is uncountable, there must be an uncountable number of equivalence class.

Each equivalence class is dense in $\mathbb{R}$ and has non-empty intersection with $[0,1]$.

By axiom of choice, there is a set P which consists of precisely one element of each equivalence class belongs to $[0,1]$.

If $x, y \in P$, are arbitrary elements of P.

then $x-y$ is irrational, for otherwise they would belong to the same equivalence class, contrary to the def of P.

We claim that P is not Lebesgue Mble.

Suppose on contrary that P is Lebesgue Mble. Then the translated sets

$$P+q = \{x+q : x\in P\}, \quad q\in Q$$

are Lebesgue measurable.

The sets $P+q$ are disjoints that is

$$(P+q)\cap(P+r) = \phi, \text{ whenever } q, r\in Q, q\neq r.$$

To see this, suppose on contrary that

$y \in (P+q) \cap (P+r)$ with $q \neq r$.

Then $y = x+q$ and $y = z + r$, for some $x, z \in P$.
Thus
$$x - z = (y-q) - (y-r) = r - q \in \mathbb{Q}$$
which implies that $x \sim z$.
Since P contains exactly one element of each equivalence class, we have $x = z$ and consequently $r = q$, a contradiction.
Hence
$$(P+q) \cap (P+r) = \phi, \quad q, r \in \mathbb{Q},\ q \neq r.$$

Next we claim that
$$[0,1] \subset \bigcup_{q \in [-1,1] \cap \mathbb{Q}} (P+q) \subset [-1,2].$$

Let $x \in [0,1]$ and let y be the representative of the equivalence class P_x belonging to P. In particular, $x \sim y$ from which it follows that $x - y \in \mathbb{Q}$.

Let $q = x - y$, since $x, y \in [0,1]$ we have $q \in [-1, 1]$ and
$$x = y + q \in P + q$$
$\Rightarrow$
$$[0,1] \subset \bigcup_{q \in [-1,1] \cap \mathbb{Q}} (P+q)$$
Now,
Since the translated set $E+q$, $q \in [-1,1] \cap \mathbb{Q}$, are disjoint and Lebesgue mble.

By countably additive and translation invariance

$$m^*\left(\bigcup_{q\in[-1,1]\cap Q}(P+q)\right)=\sum_{q\in[-1,1]\cap Q} m^*(E+q)=\sum_{q\in[-1,1]\cap Q} m^*(E)$$

which is 0 if $m^*(E)=0$ and ∞ if $m^*(E)>0$.

On the other hand

$$0<1=m^*([0,1])\le m^*\left(\bigcup_{q\in[-1,1]\cap Q}(E+q)\right)\le m^*([-1,2])=3<\infty$$

This is a contradiction and thus P cannot be Lebesgue mble.

A/2016.

Qno 8(a). ii).

Let ϕ and ψ be simple functions on D which vanishes outside a set E of finite measure and a, b are real numbers then

i) $\int a\phi+b\psi = a\int\phi+b\int\psi$.

Proof.

Let

$$\phi=\sum_{i=1}^{n} a_i \chi_{A_i}\quad,\quad \psi=\sum_{i=1}^{n} b_i \chi_{B_i}$$

Where $A_i\cap A_j=\phi$ and $B_i\cap B_j=\phi$ for $i\neq j$.

Define $E_k=\{A_i\cap B_j : i=1,2,\dots,n;\ j=1,2,3,\dots,m\}$

Then E_k are disjoint,

$$\bigcup_i A_i \subseteq \bigcup E_k \text{ and } \bigcup_i B_i \subseteq E_k.$$

Then we can write

$$\phi = \sum_{k=1}^{N} c_k \chi_{E_k}, \quad \psi = \sum_{k=1}^{N} d_k \chi_{E_k}$$

and

$$a\phi + b\psi = a\sum_{k=1}^{N} c_k \chi_{E_k} + b\sum_{k=1}^{N} d_k \chi_{E_k}$$

$$= \sum_{k=1}^{N} (ac_k + bd_k)\chi_{E_k}$$

$$\int a\phi + b\psi = \sum_{k=1}^{N} (ac_k + bd_k)\, m(E_k)$$

$$= a\sum_{k=1}^{N} c_k m(E_k) + b\sum_{k=1}^{N} d_k m(E_k)$$

$$= a\int\phi + b\int\psi$$

A/2016.

Q no 8(b)(iii)

$$\text{let } \phi(x) = \begin{cases} 2 & \text{if } x \in Q' \cap (-5,5) \\ 3 & \text{if } x \in Q \cap (-5,5) \end{cases}$$

then find the Lebesgue integration of $\int_{-5}^{5} \phi(x)\, dm$

Sol.

Let

$A = Q' \cap (-5,5)$, $B = Q \cap (-5,5)$

then

$\varphi(x) =$ ~~2 m(A)~~ $2 \cdot \chi_A + 3 \cdot \chi_B$.

$\int \phi \, dm = 2\, m(A) + 3\, m(B)$

$=$ ~~$2(0) + 3(5+5)$~~

$= 2(5+5) + 3(0)$

$= 20.$

A/2016.

Q no 8 (b) ~~etc~~

Let f be a measurable function defined on m'ble set E, then

i) f is integrable iff $|f|$ is integrable on E.

ii) If f is integrable, then $|\int f| \leq \int |f|$.

Proof.

i) Suppose that f is integrable. So by def

$\int f^+ < \infty$ and $\int f^- < \infty$

$|f|^+ = |f| = f^+ + f^-$

$|f|^- = \max\{-|f|, 0\} = 0.$

Since

$\int |f|$

and

$\int |$

$|f|$ is

Conversel

if $|f|$ is in

$\int |f|$

As

$\int |f$

So if

$\int f^+$

Hence

Altern

f is i

Since

$$\int |f|^+ = \int f^+ + f^- = \int f^+ + \int f^- < \infty.$$

and

$$\int |f|^- < \infty, \text{ so}$$

$|f|$ is integrable

Conversely

if $|f|$ is integrable, then $\int |f|^+ < \infty$ and

$$\int |f|^- < \infty.$$

As

$$\int |f|^+ = \int |f| = \int f^+ + \int f^- < \infty.$$

So it follows that

$$\int f^+ < \infty \quad \text{and} \quad \int f^- < \infty$$

Hence f is integrable.

Alternatively.

f is integrable $\Leftrightarrow \int f^+ < \infty$ and $\int f^- < \infty$

$\Leftrightarrow \int f^+ + \int f^- < \infty$

$\Leftrightarrow |f|$ is integrable

$\int \quad \int$

ii) $|\int f| = |\int f^+ - \int f^-| \leq |\int f^+| + |\int f^-|$

$= \int f^+ + \int f^- \qquad \because \int f^+, \int f^- \geq 0$

$\Rightarrow |\int f| \leq \int |f|.$

A| 2016.

A|2015 Qno 6 (b)

Qno 7(a) Prove that a continuous function defined on a measurable set is mble.

Proof.

Let f be a continuous function defined on $D \subseteq \mathbb{R}$.

For the measurability of f, it is sufficient to prove that $\{x : f(x) > \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.

Note that $\{x \in D : f(x) > \alpha\} = f^{-1}(\alpha, \infty)$.

Since f is continuous so $f^{-1}(\alpha, \infty)$ is an open subset of D

By def of relative topology on D, $\exists$ open set F in $\mathbb{R}$ such that

$f^{-1}(\alpha, \infty) = D \cap F$.

The set F being an open set is

measura

So f^-

hence

A|2015

Qno 7

function

functi

measu

Proof.

Def

1- ϕ

• Since f

So ϕ m

2- A

• If A

Now f

$\Rightarrow$

3- Coun

• If $\{G_n$

$\therefore f$

Hence

f

So

measurable.
So $f^{-1}(\alpha, \infty)$ is a measurable set and hence f is measurable function.

A/2015.

Q no 7(b) Let $f: \mathbb{R} \to \mathbb{R}$ be a measurable function and $g: \mathbb{R} \to \mathbb{R}$ a Borel measurable function. Then prove that $g \circ f$ is measurable function.

Proof.

Define $\Omega = \{G \subseteq \mathbb{R} : f^{-1}(G) \in \mathcal{M}\}$ (set of all measurable sets)

1- ϕ and $\mathbb{R} \in \Omega$

- Since $f^{-1}(\phi) = \phi$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are in $\mathcal{M}$.

So ϕ and $\mathbb{R}$ belongs to Ω.

2- $A^c \in \Omega$

- If $A \in \Omega$, then $f^{-1}(A) \in \mathcal{M}$.

Now $f^{-1}(A') = [f^{-1}(A)]' \in \mathcal{M}$.

$\Rightarrow A' \in \Omega$

3- Countable union belongs to Ω.

- If $\{G_n\}$ is a seq. of sets in Ω, then

$\therefore f^{-1}(G_n) \in \mathcal{M}$ for each n

Hence

$$f^{-1}\left(\bigcup_{n=1}^{\infty} G_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(G_n) \in \mathcal{M}.$$

So $\bigcup_{n=1}^{\infty} G_n \in \Omega$

Thus Ω is a σ-algebra on $\mathbb{R}$

To show that gof is measurable, it is sufficient to show that $(gof)^{-1}(V)$ is measurable set for any set V in $\mathbb{R}$.

Let V be any open set in $\mathbb{R}$.
Since f is measurable therefore $f^{-1}(V)$ is measurable.
This gives $V \in \Omega$.
Hence every open set in $\mathbb{R}$ is a member of Ω
Thus Ω contains the collection $\mathcal{B}$ of all Borel sets in $\mathbb{R}$ i.e, $\mathcal{B} \subset \Omega$
Now since g is Borel measurable so $g^{-1}(V)$ is also Borel set and hence

$$g^{-1}(V) \in \mathcal{B} \subseteq \Omega$$

Thus by def of Ω

$$f^{-1}(g^{-1}(V)) = (gof)^{-1}(V) \text{ is a}$$

measurable set.
Hence gof is a measurable function.

A/2015, A/2011 Q no 9(a).

Q no 8(a) Let f be a bounded function defined on $[a,b]$ and f is Reimann integrable on $[a,b]$ then show that
i) f is measurable

ii) $R\int_a^b f = \int_a^b f$

We
R
for a
As
the set
that
R
φ and
As
sup f
Thus
$R\int_a^b f$
But
So
R
Thus
sup
φ≤f
Hence

We note that,

$$R\underline{\int_a^b} f = \sup_{\phi \le f} \int \phi \quad \text{and} \quad R\overline{\int_a^b} f = \inf_{\psi \ge f} \int \psi \longrightarrow ①$$

for all step functions ϕ and ψ.

As the class of step functions is a subclass of the set of simple functions so it follows from ① that,

$$R\underline{\int_a^b} f \le \sup_{\phi \le f} \int \phi \quad \text{and}$$

$$R\overline{\int_a^b} f \ge \inf_{\psi \ge f} \int \psi \quad \text{for simple functions}$$

ϕ and ψ.

As $\phi \le \psi$ so $\int \phi \le \int \psi$ and hence

$$\sup \int \phi \le \inf \int \psi.$$

Thus

$$R\underline{\int_a^b} f \le \sup_{\phi \le f} \int \phi \le \inf_{\psi \ge f} \int \psi \le R\overline{\int_a^b} f \longrightarrow ②$$

But f is given to be Reimann integrable so

$$R\underline{\int_a^b} f = R\overline{\int_a^b} f$$

Thus

$$\sup_{\phi \le f} \int \phi = \inf_{\psi \ge f} \int \psi$$

Hence f is mble.

ii) From ①, we get

$$R\int_a^b f = \sup_{\varphi \le f} \int \varphi = \inf_{\psi \ge f} \int \psi$$

Since $m([a,b]) < \infty$ and f is measurable so,

$$\int_a^b f = \inf_{\psi \ge f} \int \psi$$

Thus

$$\int_a^b f = R\int_a^b f.$$

A/2014.

Q no 5 (b).

Show that the Cantor set C has Lebesgue measure zero.

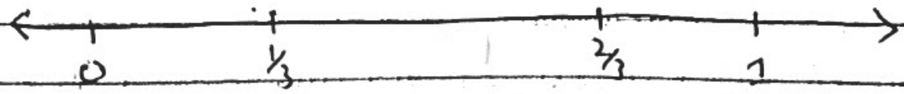
Proof.

a) Construction.

Consider the closed unit interval $[0, 1]$

One-third removal algo.

Step I $[0, 1]$



let $I_{11} = \left(\frac{1}{3}, \frac{2}{3}\right) \Rightarrow l(I_{11}) = \frac{1}{3}$.

and $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

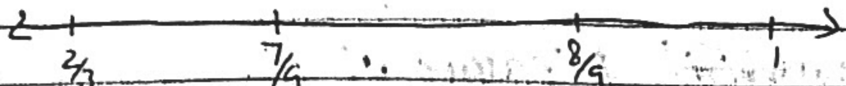
Step II.

i) $[0, 1/3]$.



let $I_{21} = \left(\frac{1}{9}, \frac{2}{9}\right) \Rightarrow l(I_{21}) = \frac{1}{9} = \frac{1}{3^2}$

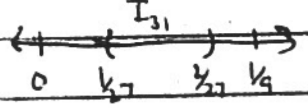
ii) $\left[\frac{2}{3}, 1\right]$.

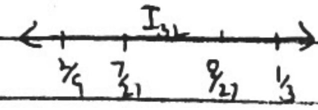


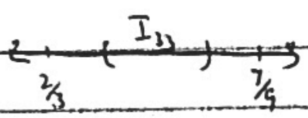
let $I_{22} = \left(\frac{7}{9}, \frac{8}{9}\right) \Rightarrow l(I_{22}) = \frac{1}{9} = \frac{1}{3^2}$

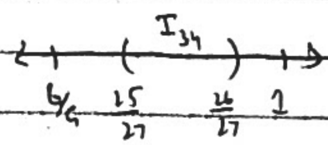
$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.

Step III.

i) 

, ii) 

iii) 

iv) 

$$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right) \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup$$

$$\left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, \frac{1}{3}\right]$$

and so on.

Define

G = (Union of all removal parts)

$$G = \bigcup_{n=1}^{\infty} \left(\bigcup_{j=1}^{2^{n-1}} I_{n,j} \right).$$

Thus G is mble

Now let

$$C = [0,1] \setminus G = \bigcap_{n=1}^{\infty} C_n$$

$$= [0,1] \cap G^c$$

Therefore C is mble

(b) ~~Measurability~~ Measure of C.

$$m(C) = m\left([0,1] \setminus G\right) = m([0,1]) - m(G)$$

First we check the measure of G.

$$m(G) = m\left(\bigcup_{n=1}^{\infty} \left(\bigcup_{j=1}^{2^{n-1}} I_{nj} \right) \right)$$

$$= \sum_{n} \sum_{j} m(I_{nj})$$

$$= m(I_{11}) + [m(I_{21}) + m(I_{22})] + [m(I_{31}) + m(I_{32}) + m(I_{33}) + m(I_{34})] + \dots$$

$m(G) =$

$\Rightarrow m($

$m(C)$

$m(C)$

Qno 7

defined

an op

if and

Proof.

open

Then

Since

$I_n =$

$$= \frac{1}{3} + \left(\frac{1}{9} + \frac{1}{9}\right) + \left(\frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27}\right) + \cdots$$

$$= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots$$

$$m(G) = \frac{1/3}{1 - 1/3} = 1$$

$\Rightarrow$ $m(G) < \infty$, so we Apply Excision property.

$$m(C) = m([0,1]) - m(G)$$

$$= 1 - 1$$

$m(C) = 0$, as required

A/2013.

Qno 7(b),

Let f be a Real valued function defined on a Measurable domain D and G is an open set in $\mathbb{R}$. Prove that f is measurable if and only if $f^{-1}(G)$ is measurable.

Proof.

Let f be measurable and G be an open set in $\mathbb{R}$.

Then $G = \bigcup_{n=1}^{\infty} I_n$ where $I_n = (a_n, b_n)$ are disjoint open intervals

Since

$I_n = (a_n, b_n) = (-\infty, b_n) \cap (a_n, \infty)$, so

$f^{-1}(I_n) = f^{-1}\{(a_n, b_n)\} = f^{-1}[(-\infty, b_n)] \cap f^{-1}[(a_n, \infty)]$.

$= \{x \in D : f(x) < b_n\} \cap \{x \in D : f(x) > a_n\}$.

Thus $f^{-1}(I_n)$ is mble for each n and so

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \text{ is mble.}$$

Conversely, suppose that

$f^{-1}(G)$ is a mble set for any open set G in $\mathbb{R}$.

In particular

$G = (\alpha, \infty)$, $\alpha \in \mathbb{R}$, f

$f^{-1}(G) = f^{-1}[(\alpha, \infty)]$

$= \{x \in D : f(x) > \alpha\}$

is a mble set. Hence f is mble.

A/2013.
Qno 7(a).

Let f be a Borel measurable function and B a Borel set. Then $f^{-1}(B)$ is a Borel set.

Proof.

Let f be a Borel mble function and B be a Borel set

Define $\Omega = \{E \subseteq \mathbb{R} : f^{-1}(E) \in \mathcal{B}\}$, which is a σ-algebra on $\mathbb{R}$.

If V is an open set, then $f^{-1}(V) \in \mathcal{B}$ $\because$ f is Borel Msble function

By def of Ω we find $V \in \Omega$.

Since $\mathcal{B}$ is the smallest σ-alogebra which contains all opensets so it follows that $\mathcal{B} \subseteq \Omega$. Hence any Borel set B is in Ω and so $f^{-1}(B) \in \mathcal{B}$

A/2012.

Qno 6 (a). Let $\{f_n\}$ be a sequence of extended real-valued mearmasble functions with same domain D. Then.

i) $\max_{1 \le i \le n} f_i$ is measurable for each n.

ii) $\min_{1 \le i \le n} f_i$ is mearurasle for each n.

Proof. Let $\{f_n\}$ be s sequence of extended real-valued msble functions with same domain D. Then let,

$$h = \max_{1 \le i \le n} f_i$$

for any $\alpha \in \mathbb{R}$, we prove that

$$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}.$$

Let $y \in \{x : h(x) > \alpha\}$

$\Rightarrow \quad h(y) > \alpha$.

Now

$h(x) = \underset{1 \le i \le n}{\text{Max}} f_i(x)$ so there exist $j \in \{1, 2, \ldots, n\}$

such that

$h(x) = f_j(x)$.

i.e., $h(y) = f_j(y) > \alpha$

$\Rightarrow \quad y \in \{x : f_j(x) > \alpha\}$

for some $j \in \{1, 2, \ldots, n\}$.

Hence $y \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$.

Conversely

Let $y \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$.

Then

$y \in \{x : f_k(x) > \alpha\}$ for some $k \in \{1, 2, \ldots, n\}$.

i.e., $f_k(y) > \alpha$.

By def of $\underset{1 \le i \le n}{\text{Max}} f_i(y) = h(y) \;\forall\; i = 1, 2, \ldots, n$.

and so $f_k(y) \le h(y) \Rightarrow h(y) > \alpha$

Thus $y \in \{x : h(x) > \alpha\}$

Hence the result

(ii) Let

$\{x : g($

as M

~~Quot~~

(ii) let $g \in \underset{1 \le i \le n}{\text{Min}} f_i$

$\{x : g(x) > \alpha\} = \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$. same proof as Max.

$1, 2, \dots, n\}$

~~Quest(b)~~